

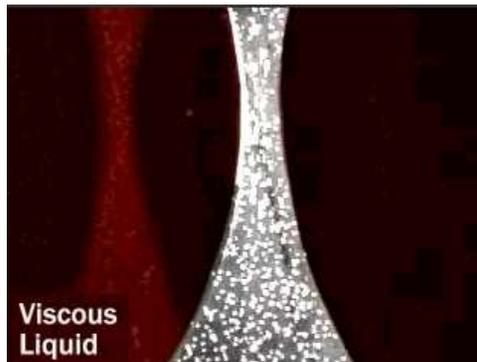
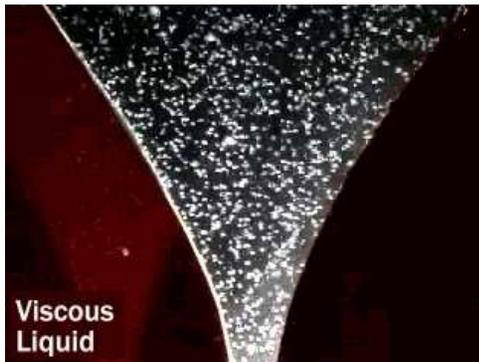
# Lecture 4 – Classification of Flows

## Applied Computational Fluid Dynamics

Instructor: André Bakker

# Classification: fluid flow vs. granular flow

- Fluid and solid particles: fluid flow vs. granular flow.
- A fluid consists of a large number of individual molecules. These could in principle be modeled as interacting solid particles.
- The interaction between adjacent salt grains and adjacent fluid parcels is quite different, however.



# Reynolds number

- The Reynolds number  $Re$  is defined as:  $Re = \rho V L / \mu$ .
- Here  $L$  is a characteristic length, and  $V$  is the velocity.
- It is a measure of the ratio between inertial forces and viscous forces.
- If  $Re \gg 1$  the flow is dominated by inertia.
- If  $Re \ll 1$  the flow is dominated by viscous effects.

# Effect of Reynolds number

Re = 0.05

Re = 10

Re = 200

Re = 3000



# Newton's second law

- For a solid mass:  $\mathbf{F} = m \cdot \mathbf{a}$

- For a continuum:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = \nabla \cdot \boldsymbol{\sigma} + f$$

Mass per volume (density)      acceleration      Force per area (stress tensor)      Force per volume (body force)

- Expressed in terms of velocity field  $\mathbf{u}(x,y,z,t)$ . In this form the momentum equation is also called Cauchy's law of motion.
- For an incompressible Newtonian fluid, this becomes:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \mu \nabla^2 \mathbf{u}$$

- Here  $p$  is the pressure and  $\mu$  is the dynamic viscosity. In this form, the momentum balance is also called the Navier-Stokes equation.

# Scaling the Navier-Stokes equation

- For unsteady, low viscosity flows it is customary to make the pressure dimensionless with  $\rho V^2$ . This results in:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \mu \nabla^2 \mathbf{u}$$

with dimensionless variables :

$$(x^*, y^*, z^*) = (x/L, y/L, z/L)$$

$$\mathbf{u}^* = \mathbf{u}/V; \quad p^* = p/\rho V^2; \quad t^* = t/(L/V)$$

this becomes :

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla p^* + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}^*$$

# Euler equation

- In the limit of  $Re \rightarrow \infty$  the stress term vanishes:

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla p^*$$

- In dimensional form, with  $\mu = 0$ , we get the Euler equations:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p$$

- The flow is then inviscid.

# Scaling the Navier-Stokes equation - viscous

- For steady state, viscous flows it is customary to make the pressure dimensionless with  $\mu V/L$ . This results in:

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u}$$

with dimensionless variables :

$$(x^*, y^*, z^*) = (x/L, y/L, z/L)$$

$$\mathbf{u}^* = \mathbf{u}/V$$

$$p^* = \frac{p}{\mu V / L}$$

this becomes :

$$\text{Re } \mathbf{u}^* \cdot \nabla \mathbf{u}^* = -\nabla p^* + \nabla^2 \mathbf{u}^*$$

In the limit of  $\text{Re} \rightarrow 0$  the convective term vanishes :

$$-\nabla p^* + \nabla^2 \mathbf{u}^* = 0$$

# Navier-Stokes and Bernoulli

- When:
  - The flow is steady:  $\partial \mathbf{u} / \partial t = 0$
  - The flow is irrotational: the vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{u} = 0$
  - The flow is inviscid:  $\mu = 0$
- And using:  $\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times \nabla \times \mathbf{u}$
- We can rewrite the Navier-Stokes equation:

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \mu \nabla^2 \mathbf{u}$$

as the Bernoulli equation:

$$\nabla \left( \frac{p}{\rho} + \frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) = 0$$

# Basic quantities

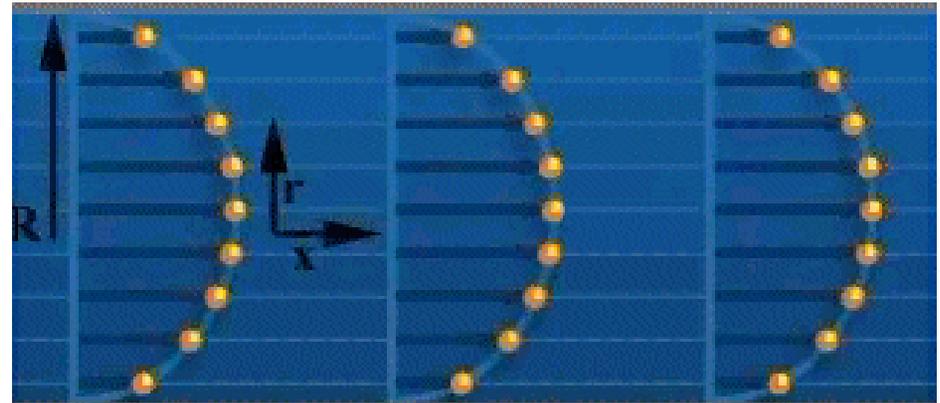
- The Navier-Stokes equations for incompressible flow involve four basic quantities:
  - Local (unsteady) acceleration.
  - Convective acceleration.
  - Pressure gradients.
  - Viscous forces.
- The ease with which solutions can be obtained and the complexity of the resulting flows often depend on which quantities are important for a given flow.

	$\rho \frac{\partial u}{\partial t}$	$\rho u \cdot \nabla u$	$= -\nabla p$	$+ \mu \nabla^2 u$
Flow	Local Acceleration	Convective Acceleration	Pressure Gradients	Viscous Forces
Pipe (steady laminar flow)			✓	✓
Plate (impulsively started)	✓			✓
Plate (boundary layer)		✓		✓
Airfoil (inviscid)		✓	✓	
Pipe (inviscid, impulsively started)	✓		✓	
Cylinder (steady viscous flow)		✓	✓	✓
Airfoil (unsteady flow)	✓	✓	✓	✓

✓ Means term is important for this flow.

# Steady laminar flow

- Steady viscous laminar flow in a horizontal pipe involves a balance between the pressure forces along the pipe and viscous forces.
- The local acceleration is zero because the flow is steady.
- The convective acceleration is zero because the velocity profiles are identical at any section along the pipe.



*Pressure Gradients and Viscous Forces*

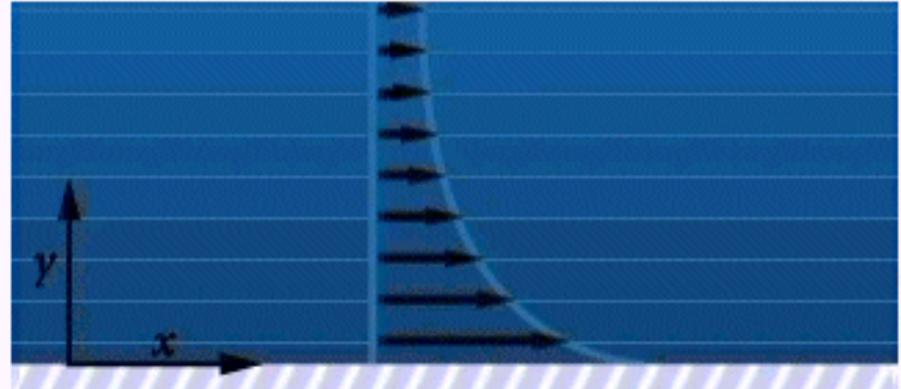
$$0 = -\nabla p + \mu \nabla^2 u$$

$$u = \left( -\frac{dp}{dx} \right) \frac{R^2}{4\mu} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]$$

*velocity profile is independent of x*

# Flow past an impulsively started flat plate

- Flow past an impulsively started flat plate of infinite length involves a balance between the local (unsteady) acceleration effects and viscous forces. Here, the development of the velocity profile is shown.
- The pressure is constant throughout the flow.
- The convective acceleration is zero because the velocity does not change in the direction of the flow, although it does change with time.



*Local Acceleration and  
Viscous Forces*

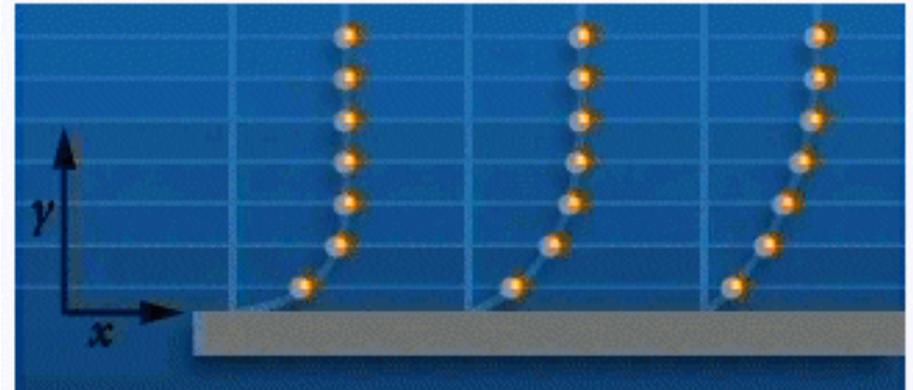
$$\frac{\partial u}{\partial t} = \nu \nabla^2 u$$

$$\nu = \mu/\rho$$

$$u = V_0 \left[ 1 - \operatorname{erf} \left( \frac{y}{\sqrt{4 \nu t}} \right) \right]$$

# Boundary layer flow along a flat plate

- Boundary layer flow along a finite flat plate involves a balance between viscous forces in the region near the plate and convective acceleration effects.
- The boundary layer thickness grows in the downstream direction.
- The local acceleration is zero because the flow is steady.



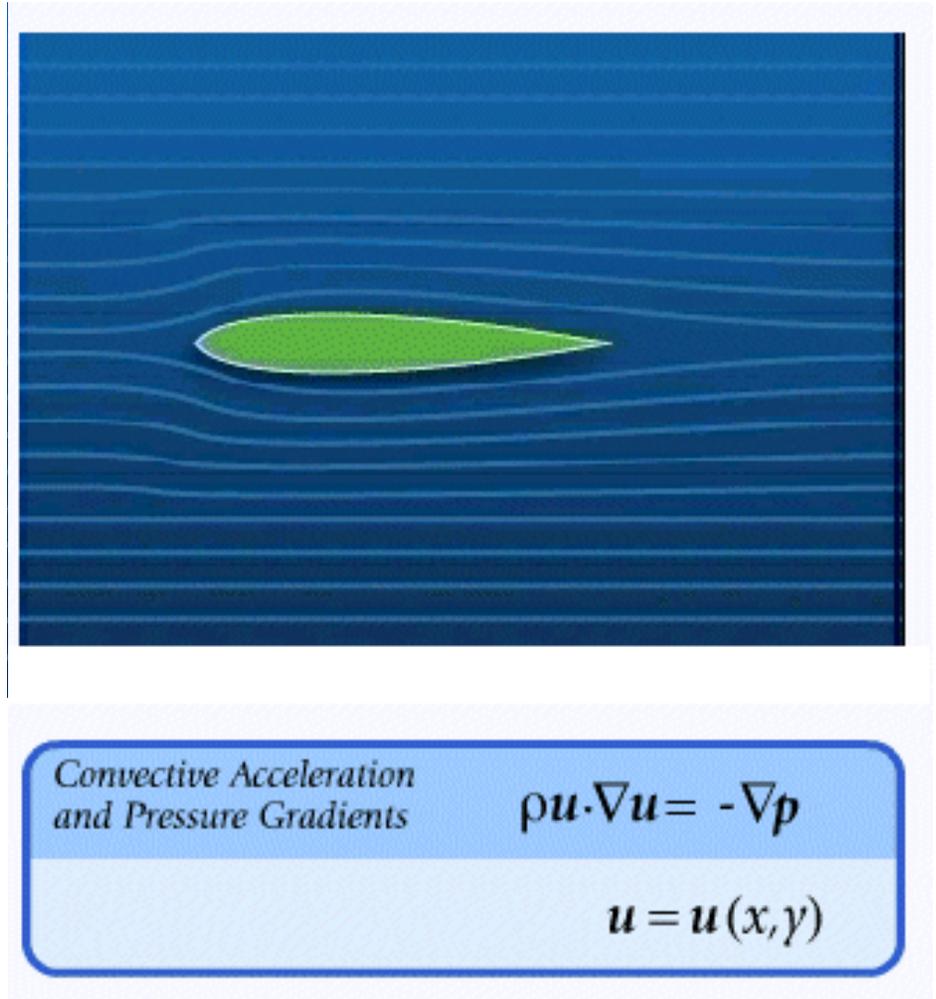
*Convective Acceleration  
and Viscous Forces*

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u}$$

$$\mathbf{u} = \mathbf{u}(x, y)$$

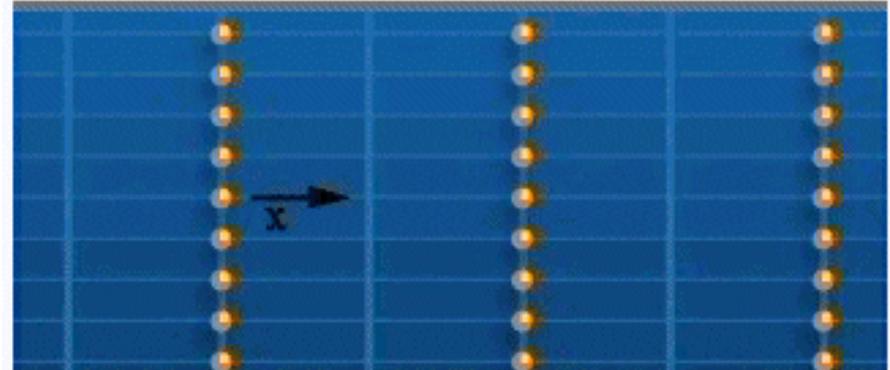
# Inviscid flow past an airfoil

- Inviscid flow past an airfoil involves a balance between pressure gradients and convective acceleration.
- Since the flow is steady, the local (unsteady) acceleration is zero.
- Since the fluid is inviscid ( $\mu=0$ ) there are no viscous forces.



# Impulsively started flow of an inviscid fluid

- Impulsively started flow of an inviscid fluid in a pipe involves a balance between local (unsteady) acceleration effects and pressure differences.
- The absence of viscous forces allows the fluid to slip along the pipe wall, producing a uniform velocity profile.
- The convective acceleration is zero because the velocity does not vary in the direction of the flow.
- The local (unsteady) acceleration is not zero since the fluid velocity at any point is a function of time.



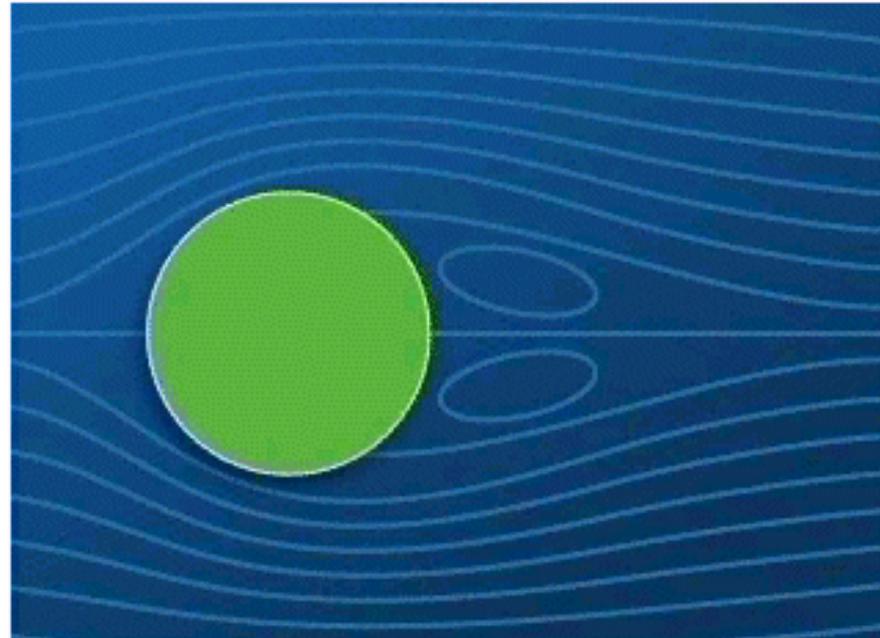
*Local Acceleration and  
Pressure Gradients*

$$\rho \frac{\partial u}{\partial t} = -\nabla p$$

$$u = u(t)$$

# Steady viscous flow past a cylinder

- Steady viscous flow past a circular cylinder involves a balance among convective acceleration, pressure gradients, and viscous forces.
- For the parameters of this flow (density, viscosity, size, and speed), the steady boundary conditions (i.e. the cylinder is stationary) give steady flow throughout.
- For other values of these parameters the flow may be unsteady.

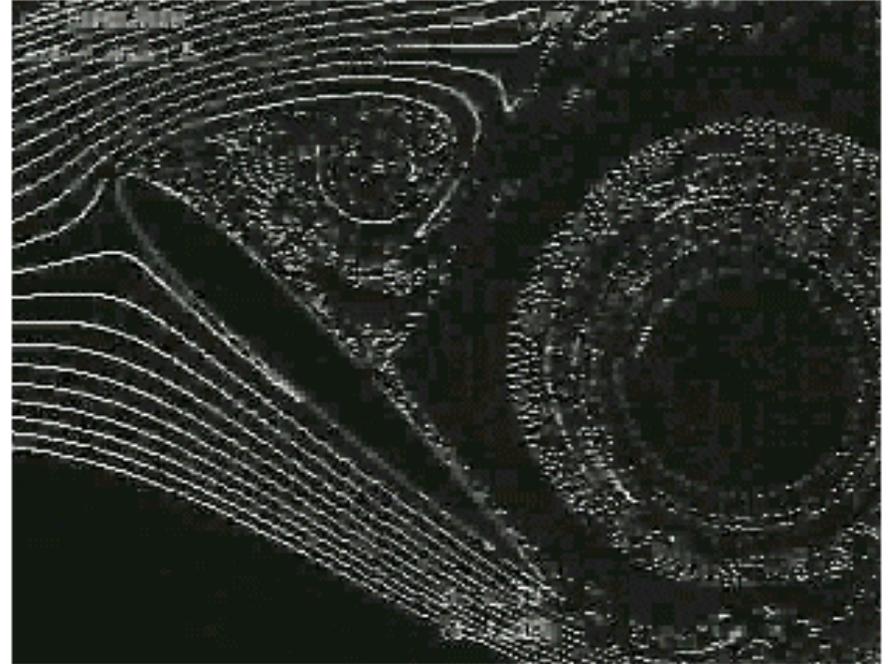


*Convective  
Acceleration,  
Pressure Gradients  
and Viscous Forces*

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u}$$

# Unsteady flow past an airfoil

- Unsteady flow past an airfoil at a large angle of attack (stalled) is governed by a balance among local acceleration, convective acceleration, pressure gradients and viscous forces.
- A wide variety of fluid mechanics phenomena often occurs in situations such as these where all of the factors in the Navier-Stokes equations are relevant.



*Local Acceleration, Convective Acceleration,  
Pressure Gradients and Viscous Forces*

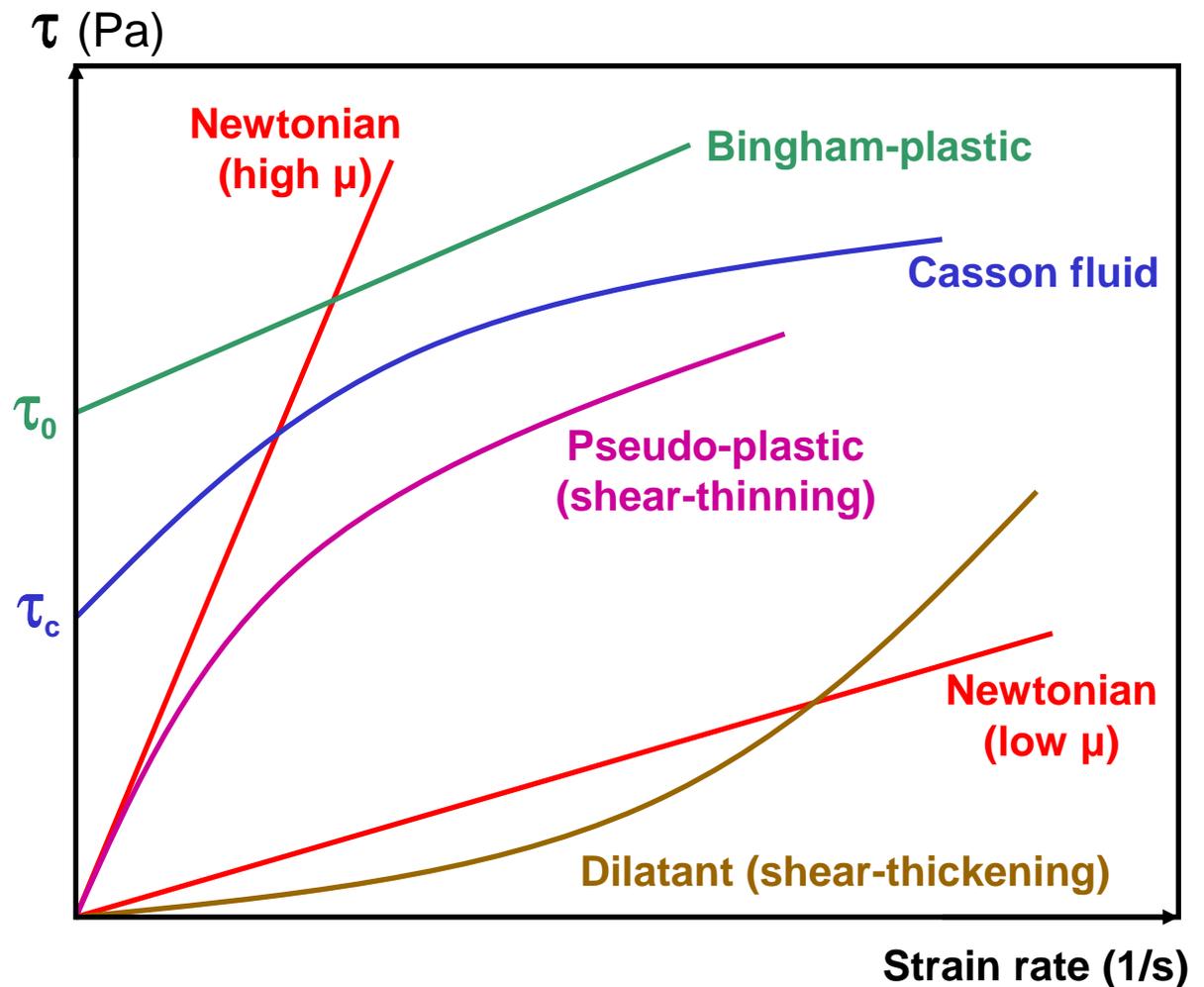
$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u}$$

# Flow classifications

- Laminar vs. turbulent flow.
  - Laminar flow: fluid particles move in smooth, layered fashion (no substantial mixing of fluid occurs).
  - Turbulent flow: fluid particles move in a chaotic, “tangled” fashion (significant mixing of fluid occurs).
- Steady vs. unsteady flow.
  - Steady flow: flow properties at any given point in space are constant in time, e.g.  $p = p(x,y,z)$ .
  - Unsteady flow: flow properties at any given point in space change with time, e.g.  $p = p(x,y,z,t)$ .

# Newtonian vs. non-Newtonian

- Newtonian fluids: water, air.
- Pseudoplastic fluids: paint, printing ink.
- Dilatant fluids: dense slurries, wet cement.
- Bingham fluids: toothpaste, clay.
- Casson fluids: blood, yogurt.
- Visco-elastic fluids: polymers (not shown in graph because viscosity is not isotropic).



# Flow classifications

- Incompressible vs. compressible flow.
  - Incompressible flow: volume of a given fluid particle does not change.
    - Implies that density is constant everywhere.
    - Essentially valid for all liquid flows.
  - Compressible flow: volume of a given fluid particle can change with position.
    - Implies that density will vary throughout the flow field.
    - Compressible flows are further classified according to the value of the Mach number (M), where.

$$M = \frac{V}{c}$$

- $M < 1$  - Subsonic.
- $M > 1$  - Supersonic.

# Flow classifications

- Single phase vs. multiphase flow.
  - Single phase flow: fluid flows without phase change (either liquid or gas).
  - Multiphase flow: multiple phases are present in the flow field (e.g. liquid-gas, liquid-solid, gas-solid).
- Homogeneous vs. heterogeneous flow.
  - Homogeneous flow: only one fluid material exists in the flow field.
  - Heterogeneous flow: multiple fluid/solid materials are present in the flow field (multi-species flows).

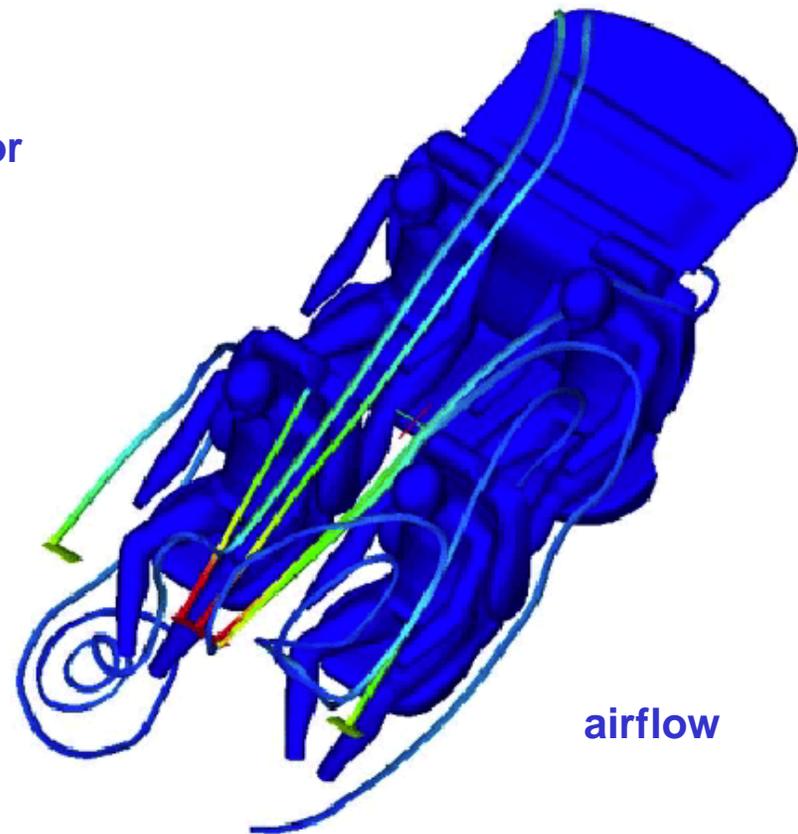
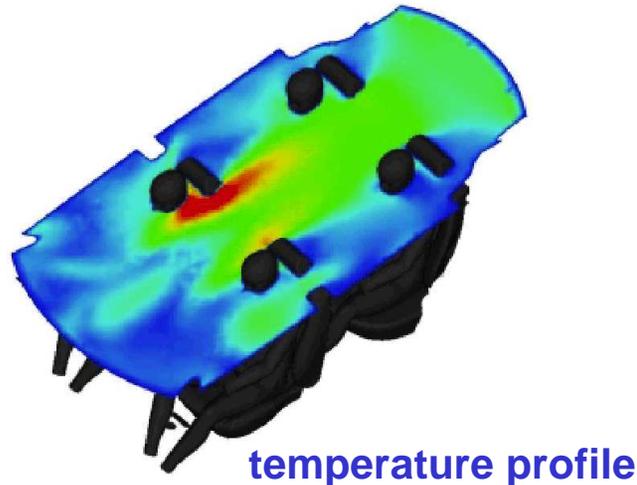
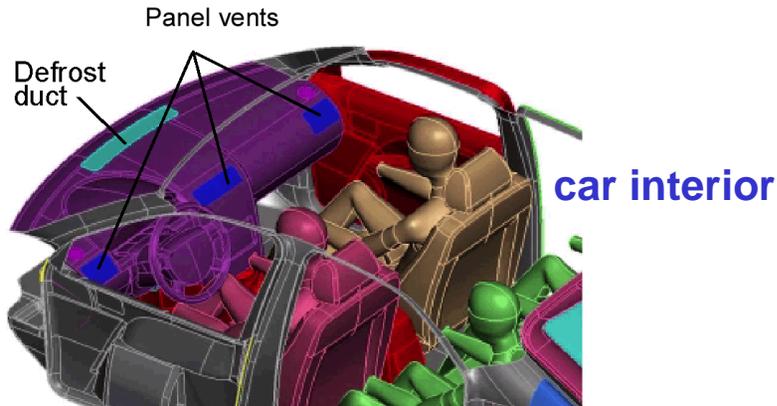
# Flow configurations - external flow

- Fluid flows over an object in an unconfined domain.
- Viscous effects are important only in the vicinity of the object.
- Away from the object, the flow is essentially inviscid.
- Examples: flows over aircraft, projectiles, ground vehicles.



# Flow configurations - internal flow

- Fluid flow is confined by walls, partitions, and other boundaries.
- Viscous effects extend across the entire domain.
- Examples: flows in pipes, ducts, diffusers, enclosures, nozzles.



# Classification of partial differential equations

- A general partial differential equation in coordinates  $x$  and  $y$ :

$$a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2} + d \frac{\partial \phi}{\partial x} + e \frac{\partial \phi}{\partial y} + f \phi + g = 0$$

- Characterization depends on the roots of the higher order (here second order) terms:
  - $(b^2 - 4ac) > 0$  then the equation is called hyperbolic.
  - $(b^2 - 4ac) = 0$  then the equation is called parabolic.
  - $(b^2 - 4ac) < 0$  then the equation is called elliptic.
- Note: if  $a$ ,  $b$ , and  $c$  themselves depend on  $x$  and  $y$ , the equations may be of different type, depending on the location in  $x$ - $y$  space. In that case the equations are of *mixed* type.

# Origin of the terms

- The origin of the terms “elliptic,” “parabolic,” or “hyperbolic” used to label these equations is simply a direct analogy with the case for conic sections.
- The general equation for a conic section from analytic geometry is:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where if.

- $(b^2-4ac) > 0$  the conic is a hyperbola.
- $(b^2-4ac) = 0$  the conic is a parabola.
- $(b^2-4ac) < 0$  the conic is an ellipse.

# Elliptic problems

- Elliptic equations are characteristic of equilibrium problems, this includes many (but not all) steady state flows.
- Examples are potential flow, the steady state temperature distribution in a rod of solid material, and equilibrium stress distributions in solid objects under applied loads.
- For potential flows the velocity is expressed in terms of a velocity potential:  $\mathbf{u}=\nabla\phi$ . Because the flow is incompressible,  $\nabla\cdot\mathbf{u}=0$ , which results in  $\nabla^2\phi=0$ . This is also known as Laplace's equation:

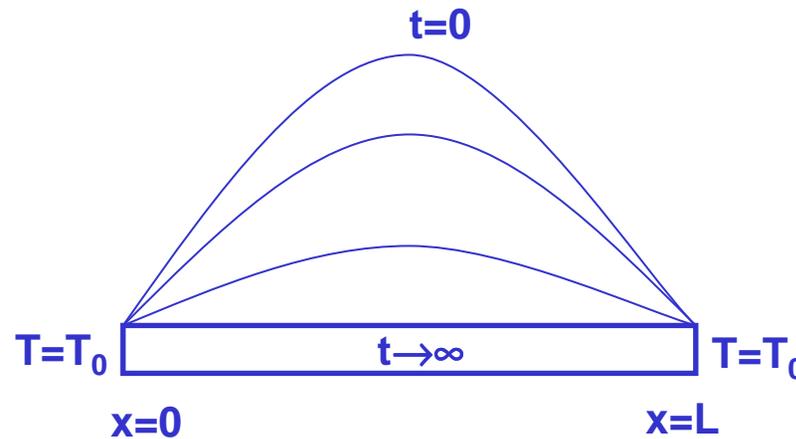
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

- The solution depends solely on the boundary conditions. This is also known as a boundary value problem.
- A disturbance in the interior of the solution affects the solution everywhere else. The disturbance signals travel in all directions.
- As a result, solutions are always smooth, even when boundary conditions are discontinuous. This makes numerical solution easier!

# Parabolic problems

- Parabolic equations describe marching problems. This includes time dependent problems which involve significant amounts of dissipation. Examples are unsteady viscous flows and unsteady heat conduction. Steady viscous boundary layer flow is also parabolic (march along streamline, not in time).
- An example is the transient temperature distribution in a cooling down rod:

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$



- The temperature depends on both the initial and boundary conditions. This is also called an initial-boundary-value problem.
- Disturbances can only affect solutions at a later time.
- Dissipation ensures that the solution is always smooth.

# Hyperbolic problems

- Hyperbolic equations are typical of marching problems with negligible dissipation.

- An example is the wave equation: 
$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

- This describes the transverse displacement of a string during small amplitude vibrations. If  $y$  is the displacement,  $x$  the coordinate along the string, and  $a$  the initial amplitude, the solution is:

$$y(x, t) = a \cos\left(\frac{\pi ct}{L}\right) \sin\left(\frac{\pi x}{L}\right)$$

- Note that the amplitude is independent of time, i.e. there is no dissipation.
- Hyperbolic problems can have discontinuous solutions.
- Disturbances may affect only a limited region in space. This is called the *zone of influence*. Disturbances propagate at the wave speed  $c$ .
- Local solutions may only depend on initial conditions in the *domain of dependence*.
- Examples of flows governed by hyperbolic equations are shockwaves in transonic and supersonic flows.

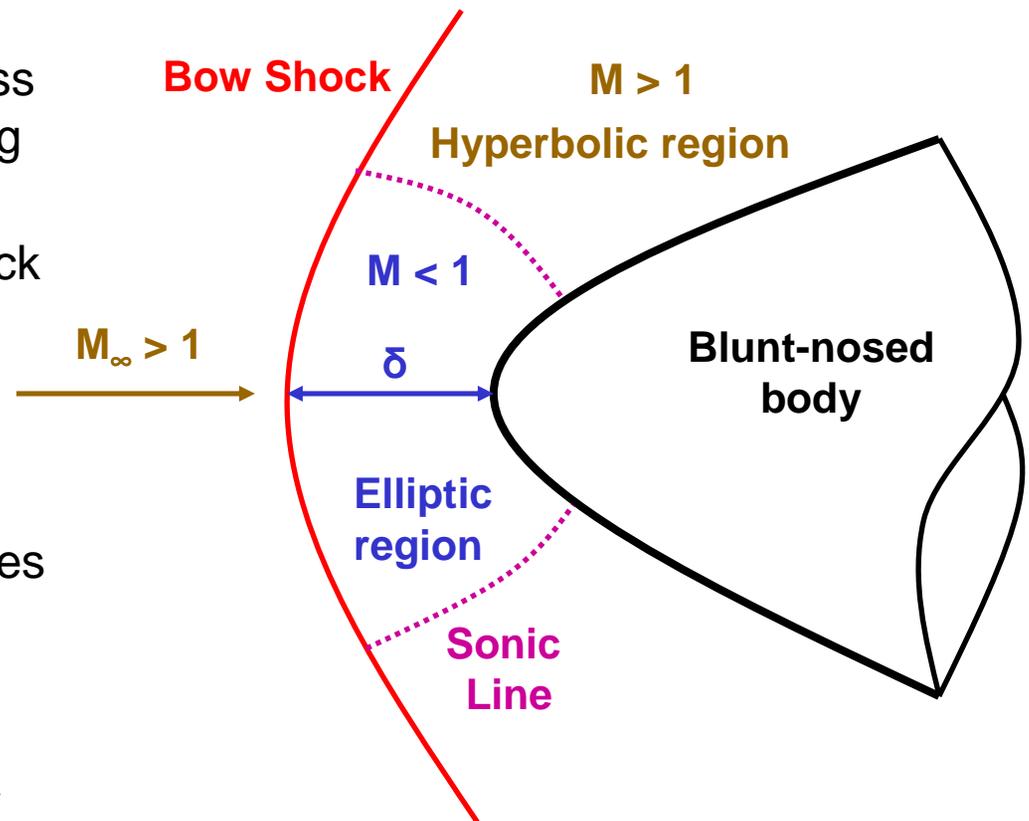
# Classification of fluid flow equations

	<b>Steady Flow</b>	<b>Unsteady Flow</b>
<i>Viscous flow</i>	Elliptic	Parabolic
<i>Inviscid flow</i>	M < 1 (subsonic) Elliptic	Hyperbolic
	M > 1 (supersonic) Hyperbolic	Hyperbolic
<i>Thin shear layers</i>	Parabolic	Parabolic

- For inviscid flows at  $M < 1$ , pressure disturbances travel faster than the flow speed. If  $M > 1$ , pressure disturbances can not travel upstream. Limited zone of influence is a characteristic of hyperbolic problems.
- In thin shear layer flows, velocity derivatives in flow direction are much smaller than those in the cross flow direction. Equations then effectively contain only one (second order) term and become parabolic. Also the case for other strongly directional flows such as fully developed duct flow and jets.

# Example: the blunt-nosed body

- Blunt-nosed body designs are used for supersonic and hypersonic speeds (e.g. Apollo capsules and spaceshuttle) because they are less susceptible to aerodynamic heating than sharp nosed bodies.
- There is a strong, curved bow shock wave, detached from the nose by the shock detachment distance  $\delta$ .
- Calculating this flow field was a major challenge during the 1950s and 1960s because of the difficulties involved in solving for a flow field that is elliptic in one region and hyperbolic in others.
- Today's CFD solvers can routinely handle such problems, provided that the flow is calculated as being transient.



# Initial and boundary conditions

- Initial conditions for unsteady flows:
  - Everywhere in the solution region  $\rho$ ,  $\mathbf{u}$  and  $T$  must be given at time  $t=0$ .
- Typical boundary conditions for unsteady and steady flows:
  - On solid walls:
    - $\mathbf{u} = \mathbf{u}_{\text{wall}}$  (no-slip condition).
    - $T=T_{\text{wall}}$  (fixed temperature) or  $k\partial T/\partial n=-q_{\text{wall}}$  (fixed heat flux).
  - On fluid boundaries.
    - For most flows, inlet:  $\rho$ ,  $\mathbf{u}$ , and  $T$  must be known as a function of position. For external flows (flows around objects) and inviscid subsonic flows, inlet boundary conditions may be different.
    - For most flows, outlet:  $-p+\mu \partial u_n/\partial n=F_n$  and  $\mu \partial u_t/\partial n=F_n$  (stress continuity).  $F$  is the given surface stress. For fully developed flow  $F_n=-p$  ( $\partial u_n/\partial n =0$ ) and  $F_t=0$ . For inviscid supersonic flows, outlet conditions may be different.
- A more detailed discussion of boundary conditions will be given later on.

# Summary

- Fluid flows can be classified in a variety of ways:
  - Internal vs. external.
  - Laminar vs. turbulent.
  - Compressible vs. incompressible.
  - Steady vs. unsteady.
  - Supersonic vs. transonic vs. subsonic.
  - Single-phase vs. multiphase.
  - Elliptic vs. parabolic vs. hyperbolic.