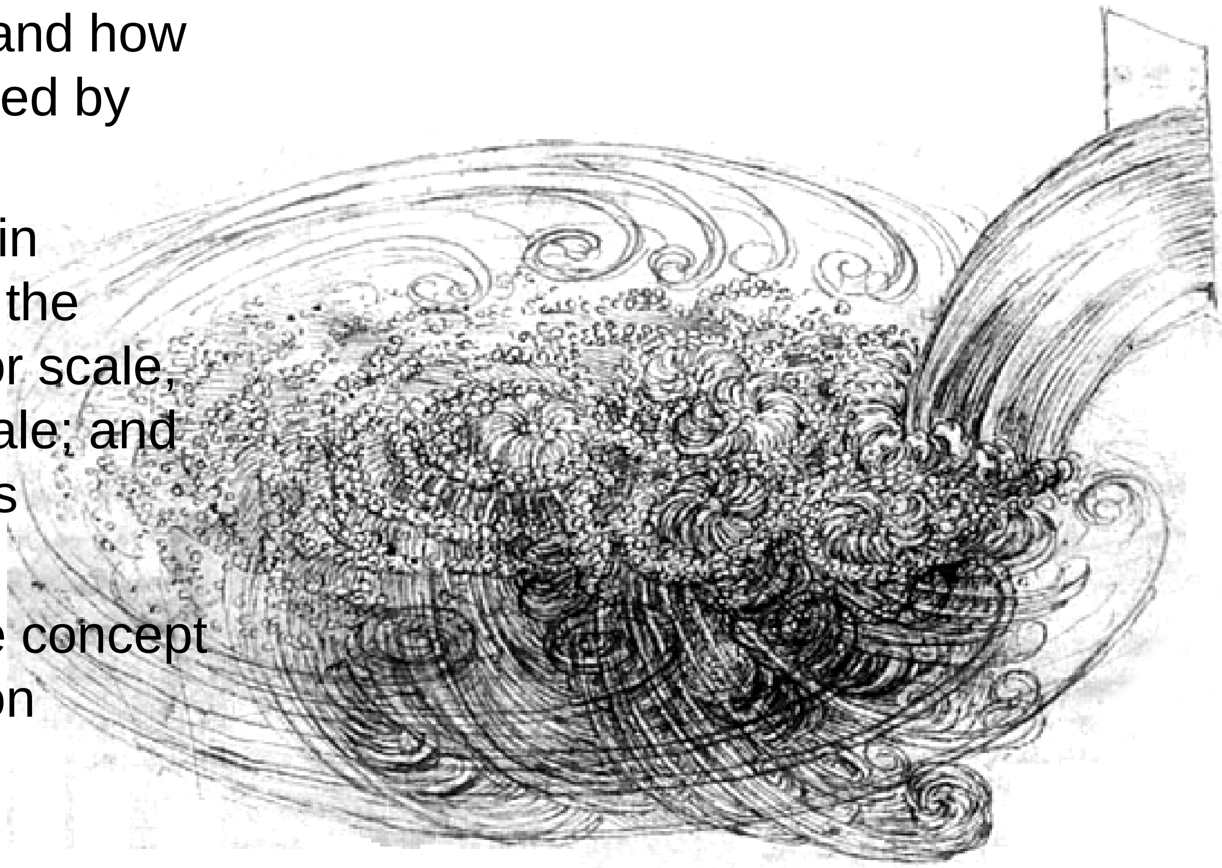
Lecture 9 - Kolmogorov's Theory

Applied Computational Fluid Dynamics

Instructor: André Bakker

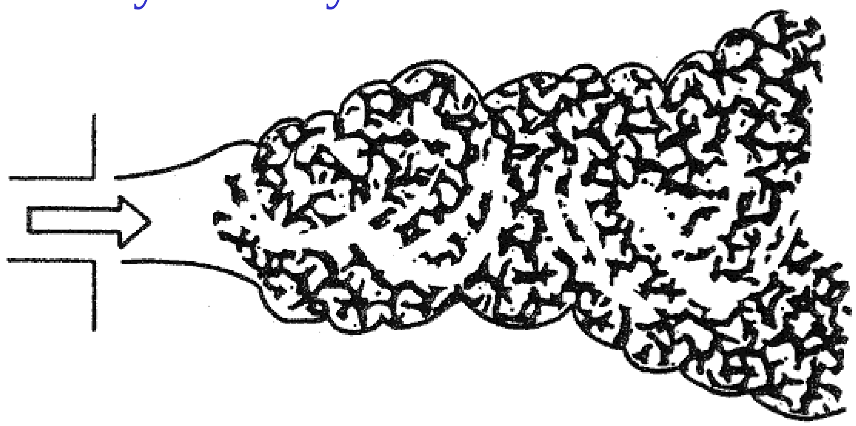
Eddy size

- Kolmogorov's theory describes how energy is transferred from larger to smaller eddies; how much energy is contained by eddies of a given size; and how much energy is dissipated by eddies of each size.
- We will derive three main turbulent length scales: the integral scale, the Taylor scale, and the Kolmogorov scale; and corresponding Reynolds numbers.
- We will also discuss the concept of energy and dissipation spectra.

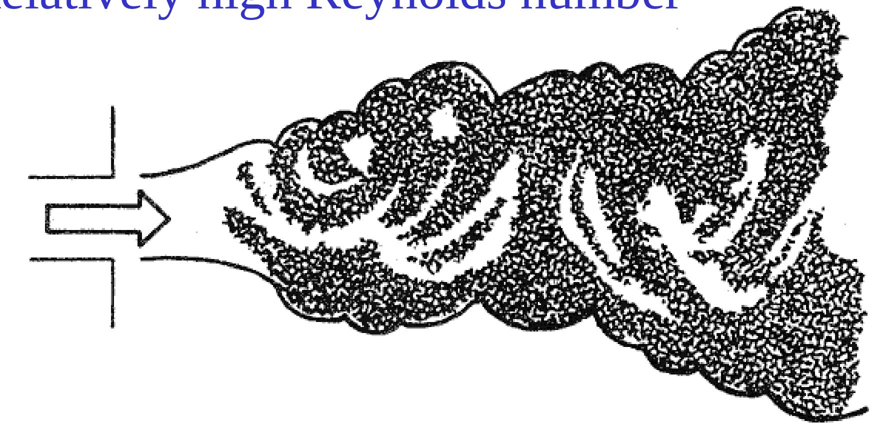


Jets at two different Reynolds numbers

Relatively low Reynolds number



Relatively high Reynolds number



Source: Tennekes & Lumley. Page 22.

Turbulent eddies

- Consider fully turbulent flow at high Reynolds number Re=UL/ν.
- Turbulence can be considered to consist of eddies of different sizes.
- An 'eddy' preludes precise definition, but it is conceived to be a turbulent motion, localized over a region of size l, that is at least moderately coherent over this region.
- The region occupied by a larger eddy can also contain smaller eddies.
- Eddies of size l have a characteristic velocity $u(l)$ and timescale $\tau(l) \equiv l/u(l)$.
- Eddies in the largest size range are characterized by the lengthscale l_0 which is comparable to the flow length scale L.
- Their characteristic velocity $u_0 \equiv u(l_0)$ is on the order of the r.m.s. turbulence intensity $u' \equiv (2k/3)^{1/2}$ which is comparable to U.
- Here the turbulent kinetic energy is defined as: $k = \frac{1}{2} < u_i u_i > = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$
- The Reynolds number of these eddies $Re_0 \equiv u_0 l_0 / \nu$ is therefore large (comparable to Re) and the direct effects of viscosity on these eddies are negligibly small.

Integral scale

- We can derive an estimate of the lengthscale l_0 of the larger eddies based on the following:
  - Eddies of size l_0 have a characteristic velocity u_0 and timescale $\tau_0 \equiv l_0/u_0$
  - Their characteristic velocity $u_0 \equiv u(l_0)$ is on the order of the r.m.s. turbulence intensity $u' \equiv (2k/3)^{1/2}$
  - Assume that energy of eddy with velocity scale u_0 is dissipated in time τ_0
- We can then derive the following equation for this length scale:

$$l_0 \propto \frac{k^{3/2}}{\varepsilon}$$

- Here, ε(m^2/s^3) is the energy dissipation rate. The proportionality constant is of the order one. This lengthscale is usually referred to as the ***integral scale*** of turbulence.
- The Reynolds number associated with these large eddies is referred to as the turbulence Reynolds number Re_L, which is defined as:

$$\mathrm{Re}_L = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\varepsilon \nu}$$

Energy transfer

- The large eddies are unstable and break up, transferring their energy to somewhat smaller eddies.
- These smaller eddies undergo a similar break-up process and transfer their energy to yet smaller eddies.
- This *energy cascade* – in which energy is transferred to successively smaller and smaller eddies – continues until the Reynolds number $\mathrm{Re}(l) \equiv u(l)l/\nu$ is sufficiently small that the eddy motion is stable, and molecular viscosity is effective in dissipating the kinetic energy.
- At these small scales, the kinetic energy of turbulence is converted into heat.

Richardson

- L.F. Richardson ("Weather Prediction by Numerical Process." Cambridge University Press, 1922) summarized this in the following often cited verse:

Big whirls have little whirls
Which feed on their velocity;
And little whirls have lesser whirls,
And so on to viscosity
in the molecular sense.

Dissipation

- Note that dissipation takes place at the end of the sequence of processes.
- The rate of dissipation ε is determined, therefore by the first process in the sequence, which is the transfer of energy from the largest eddies.
- These eddies have energy of order u_0^2 and timescale $\tau_0 = l_0/u_0$ so the rate of transfer of energy can be supposed to scale as

  $u_0^2/\tau_0 = u_0^3/l_0$
- Consequently, consistent with experimental observations in free shear flows, this picture of the energy cascade indicates that ε is proportional to u_0^3/l_0 independent of ν (at high Reynolds numbers).

Kolmogorov’s theory

- Many questions remain unanswered.
  - What is the size of the smallest eddies that are responsible for dissipating the energy?
  - As l decreases, do the characteristic velocity and timescales $u(l)$ and $\tau(l)$ increase, decrease, or stay the same? The assumed decrease of the Reynolds number $u_0 l_0/\nu$ by itself is not sufficient to determine these trends.
- These and others are answered by Kolmogorov’s theory of turbulence (1941, see Pope (2000)).
- Kolmogorov’s theory is based on three important hypotheses combined with dimensional arguments and experimental observations.

Kolmogorov's hypothesis of local isotropy

- For homogenous turbulence, the turbulent kinetic energy *k* is the same everywhere. For isotropic turbulence the eddies also behave the same in all directions: $\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$
- Kolmogorov argued that the directional biases of the large scales are lost in the chaotic scale-reduction process as energy is transferred to successively smaller eddies.
- Hence Kolmogorov's hypothesis of local isotropy states that *at sufficiently high Reynolds numbers, the small-scale turbulent motions ($l << l_0$) are statistically isotropic.*
- Here, the term local isotropy means isotropy at small scales. Large scale turbulence may still be anisotropic.
- l_{EI} is the length scale that forms the demarcation between the large scale anisotropic eddies ($l > l_{EI}$) and the small scale isotropic eddies ($l < l_{EI}$). For many high Reynolds number flows l_{EI} can be estimated as $l_{EI} \approx l_0/6$.

Kolmogorov's first similarity hypothesis

- Kolmogorov also argued that not only does the directional information get lost as the energy passes down the cascade, but that all information about the geometry of the eddies gets lost also.
- As a result, the statistics of the small-scale motions are universal: they are *similar* in every high Reynolds number turbulent flow, independent of the mean flow field and the boundary conditions.
- These small scale eddies depend on the rate T_{EI} at which they receive energy from the larger scales (which is approximately equal to the dissipation rate ε) and the viscous dissipation, which is related to the kinematic viscosity ν.
- Kolmogorov's first similarity hypothesis states *that in every turbulent flow at sufficiently high Reynolds number, the statistics of the small scale motions ($l<l_{EI}$) have a universal form that is uniquely determined by ε and ν.*

Kolmogorov scales

- Given the two parameters ε and ν we can form the following unique length, velocity, and time scales:

$$\textit{length scale}: \quad \eta = (\nu^3 / \varepsilon)^{1/4}$$

$$\textit{velocity scale}: \quad u_\eta = (\varepsilon\nu)^{1/4}$$

$$\textit{time scale}: \quad \tau_\eta = (\nu / \varepsilon)^{1/2}$$

$$(u_\eta / \eta) = 1/\tau_\eta$$

$$\mathrm{Re}_\eta = \eta u_\eta / \nu = 1$$

- These scales are indicative of the smallest eddies present in the flow, the scale at which the energy is dissipated.
- Note that the fact that the Kolmogorov Reynolds number Re_η of the small eddies is 1, is consistent with the notion that the cascade proceeds to smaller and smaller scales until the Reynolds number is small enough for dissipation to be effective.

Kolmogorov scales - derivation

$$\tau_\eta = \eta / u_\eta \quad [1]$$

$$\mathrm{Re}_\eta = \frac{\eta\, u_\eta}{\nu} = 1 \quad \Rightarrow \quad u_\eta = \frac{\nu}{\eta} \quad [2]$$

$$\varepsilon = \frac{u_\eta^2}{\tau_\eta} \sim \frac{u_\eta^2 . u_\eta}{\eta} \sim \frac{u_\eta^3}{\eta} \Rightarrow u_\eta \sim (\varepsilon\,\eta)^{1/3} \quad [3]$$

$$[2] \wedge [3] \Rightarrow \eta = \frac{\nu}{\varepsilon^{1/3}\,\eta^{1/3}} \Rightarrow \eta^{4/3} = \frac{\nu}{\varepsilon^{1/3}} \Rightarrow$$

$$\Rightarrow \textit{length scale} \quad \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \quad [4]$$

$$\varepsilon \sim \frac{u_\eta^3}{\eta} \wedge \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \Rightarrow \textit{velocity scale}: \quad u_\eta = (\varepsilon\nu)^{1/4} \quad [5]$$

$$[1] \wedge [4] \wedge [5] \Rightarrow \textit{time scale}: \quad \tau_\eta = (\nu / \varepsilon)^{1/2} \quad [6]$$

Universal equilibrium range

- The size range ($l<l_{EI}$) is referred to as the universal equilibrium range.
- In this range, the timescales $l/u(l)$ are small compared to l_0/u_0 so that the small eddies can adapt quickly to maintain dynamic equilibrium with the energy transfer rate T_{EI} imposed by the large eddies.
- On these scales all high Reynolds number flow fields are statistically identical if the flow fields are scaled by the Kolmogorov scales.

Ratio between large and small scales

- When we use the relationship $l_0 \sim k^{3/2}/\varepsilon$ and substitute it in the equations for the Kolmogorov scales, we can calculate the ratios between the small scale and large scale eddies.

$$\eta / l_0 \sim \mathrm{Re}_L^{-3/4}$$

$$u_\eta / u_0 \sim \mathrm{Re}_L^{-1/4}$$

$$\tau_\eta / \tau_0 \sim \mathrm{Re}_L^{-1/2}$$

- As expected, at high Reynolds numbers, the velocity and timescales of the smallest eddies are small compared to those of the largest eddies.
- Since η/l_0 decreases with increasing Reynolds number, at high Reynolds number there will be a range of intermediate scales l which is small compared to l_0 and large compared with η.

Kolmogorov's second similarity hypothesis

- Because the Reynolds number of the intermediate scales l is relatively large, they will not be affected by the viscosity ν.
- Based on that, Kolmogorov's second similarity hypothesis states that *in every turbulent flow at sufficiently high Reynolds number, the statistics of the motions of scale l in the range* $l_0 >> l >> \eta$ *have a universal form that is uniquely determined by* ε *independent of* ν.
- We introduce a new length scale l_{DI}, (with $l_{DI} \approx 60\eta$ for many turbulent high Reynolds number flows) so that this range can be written as $l_{EI} > l > l_{DI}$
- This length scale splits the universal equilibrium range into two subranges:
  - The inertial subrange ($l_{EI} > l > l_{DI}$) where motions are determined by inertial effects and viscous effects are negligible.
  - The dissipation range ($l < l_{DI}$) where motions experience viscous effects.

Eddy sizes in the inertial subrange

- For eddies in the inertial subrange of size l, using:

$$\varepsilon = u_\eta^3 / \eta = \eta^2 / \tau_\eta^3$$

and the previously shown relationships between the turbulent Reynolds number and various scales, velocity scales and timescales can be formed from ε and l:

$$u(l) = (\varepsilon l)^{1/3} = u_\eta (l/\eta)^{1/3} \sim u_0 (l/l_0)^{1/3}$$

$$\tau(l) = (l^2/\varepsilon)^{1/3} = \tau_\eta (l/\eta)^{2/3} \sim \tau_0 (l/l_0)^{2/3}$$

- A consequence, then, of the second similarity hypothesis is that in the inertial subrange the velocity scales and timescales $u(l)$ and $\tau(l)$ decrease as l decreases.

Taylor microscale

- The energy dissipation rate ε is given by the following equation, which comes from the analytically derived conservation equation for turbulent kinetic energy:

$$\overline{s_{ij}s_{ij}} \gg S_{ij}S_{ij}; \quad \varepsilon = 2\nu\overline{s_{ij}s_{ij}}\,; \quad s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- The lower case indicates the fluctuating components. The dissipation rate depends on the viscosity and velocity gradients ("shear") in the turbulent eddies.
- Working out this equation further for isotropic turbulence (mainly bookkeeping for all the terms) results in:

$$\varepsilon = 15\nu\ \overline{(\partial u_1 / \partial x_1)^2}$$

- We can now define the Taylor microscale λ as follows:

$$\overline{(\partial u_1 / \partial x_1)^2} \equiv \overline{u_1^2} / \lambda^2 = u'^2 / \lambda^2$$

Taylor microscale - continued

- This then results in the following relationship for the Taylor microscale λ:

$$\varepsilon = 15\nu u'^2/\lambda^2$$

- From $k = (1/2)(u'^2+v'^2+w'^2)$ we can derive $k = (2/3)u'^2$, and:

$$\lambda \approx (10\nu k/\varepsilon)^{1/2}$$

- The Taylor microscale falls in between the large scale eddies and the small scale eddies, which can be seen by calculating the ratios between λ and l_0 and η:

$$\lambda/l_0 = \sqrt{10}\,\mathrm{Re}_L^{-1/2}$$

$$\eta/l_0 = \mathrm{Re}_L^{-3/4}$$

$$\lambda/\eta = \sqrt{10}\,\mathrm{Re}_L^{1/4}$$

$$\lambda = \sqrt{10}\,\eta^{2/3}\,l_0^{1/3}$$

Taylor-scale Reynolds number

- A commonly used quantity in the characterization of turbulence is the Taylor-scale Reynolds number R_λ.
- This is based on the length scale λ and the corresponding velocity scale:

$$R_\lambda = u'\lambda/\nu$$

- R_λ can be related to the turbulence Reynolds number as follows:

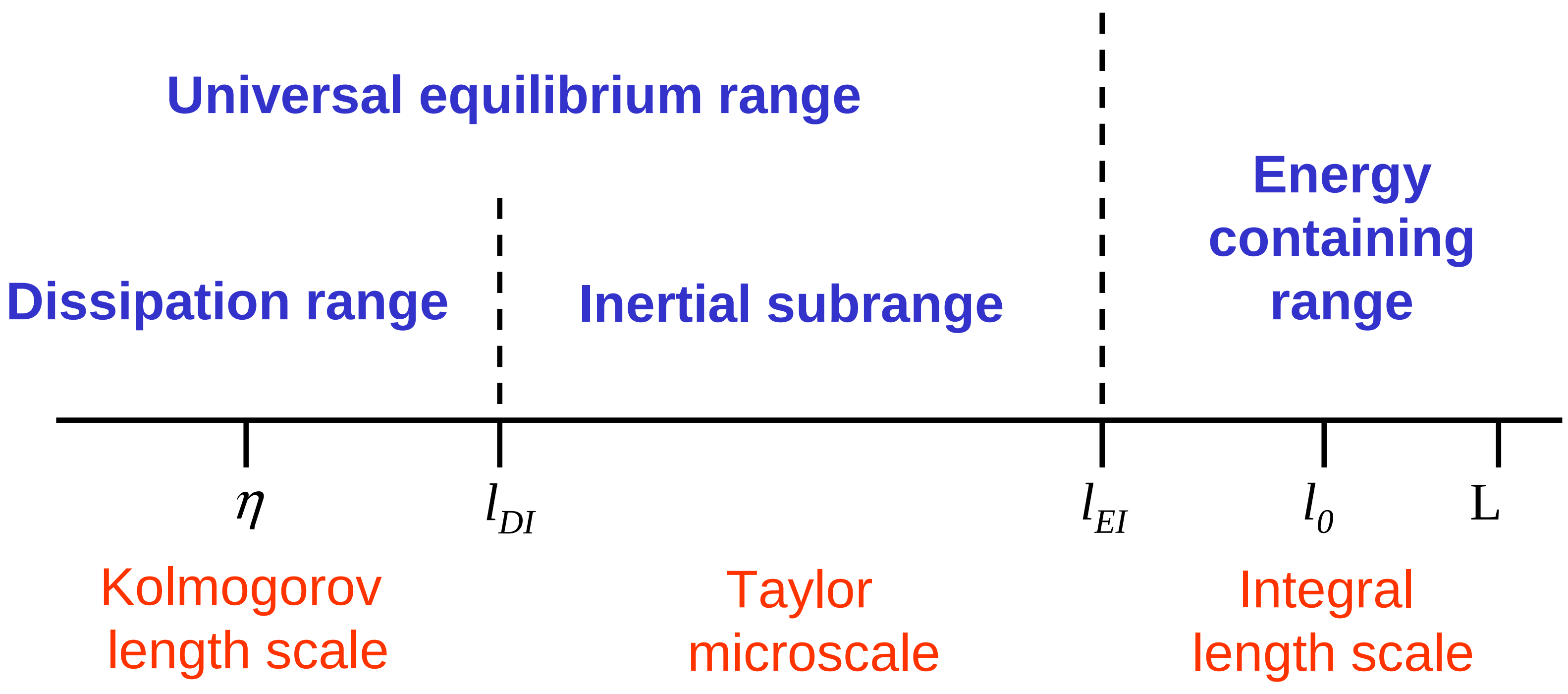
$$R_\lambda = \left(\tfrac{20}{3}\mathrm{Re}_L\right)^{1/2}$$

- We can also relate the timescale of the eddies of length scale λ to the Kolmogorov timescale:

$$\lambda/u' = (15\nu/\varepsilon)^{1/2} = \sqrt{15}\,\tau_\eta$$

Eddy sizes

- The bulk of the energy is contained in the larger eddies in the size range $l_{EI} = l_0/6 < l < 6l_0$, which is therefore called the energy-containing range.
- The suffixes EI and DI indicate that l_{EI} is the demarcation line between energy (E) and inertial (I) ranges, as l_{DI} is that between the dissipation (D) and inertial (I) ranges.



Taylor scales

- The eddy size in the inertial subrange is given by the Taylor microscale λ:

$$\lambda \approx (10 \nu k / \varepsilon)^{1/2}$$

- A commonly used quantity in the characterization of turbulence is the Taylor-scale Reynolds number R_λ.
- This is based on the length scale λ and the corresponding velocity scale:

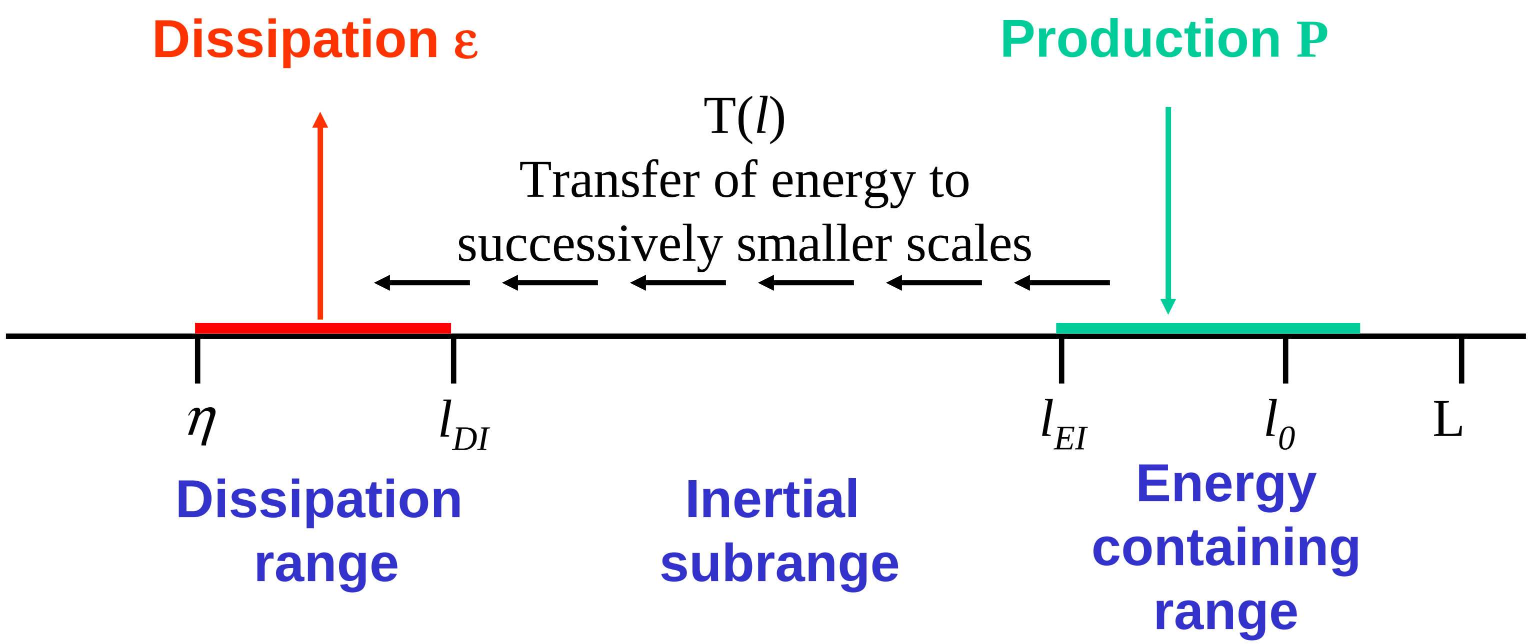
$$R_\lambda = u' \lambda / \nu$$

- R_λ can be related to the turbulence Reynolds number as follows:

$$R_\lambda = \left(\tfrac{20}{3} \mathrm{Re}_L \right)^{1/2}$$

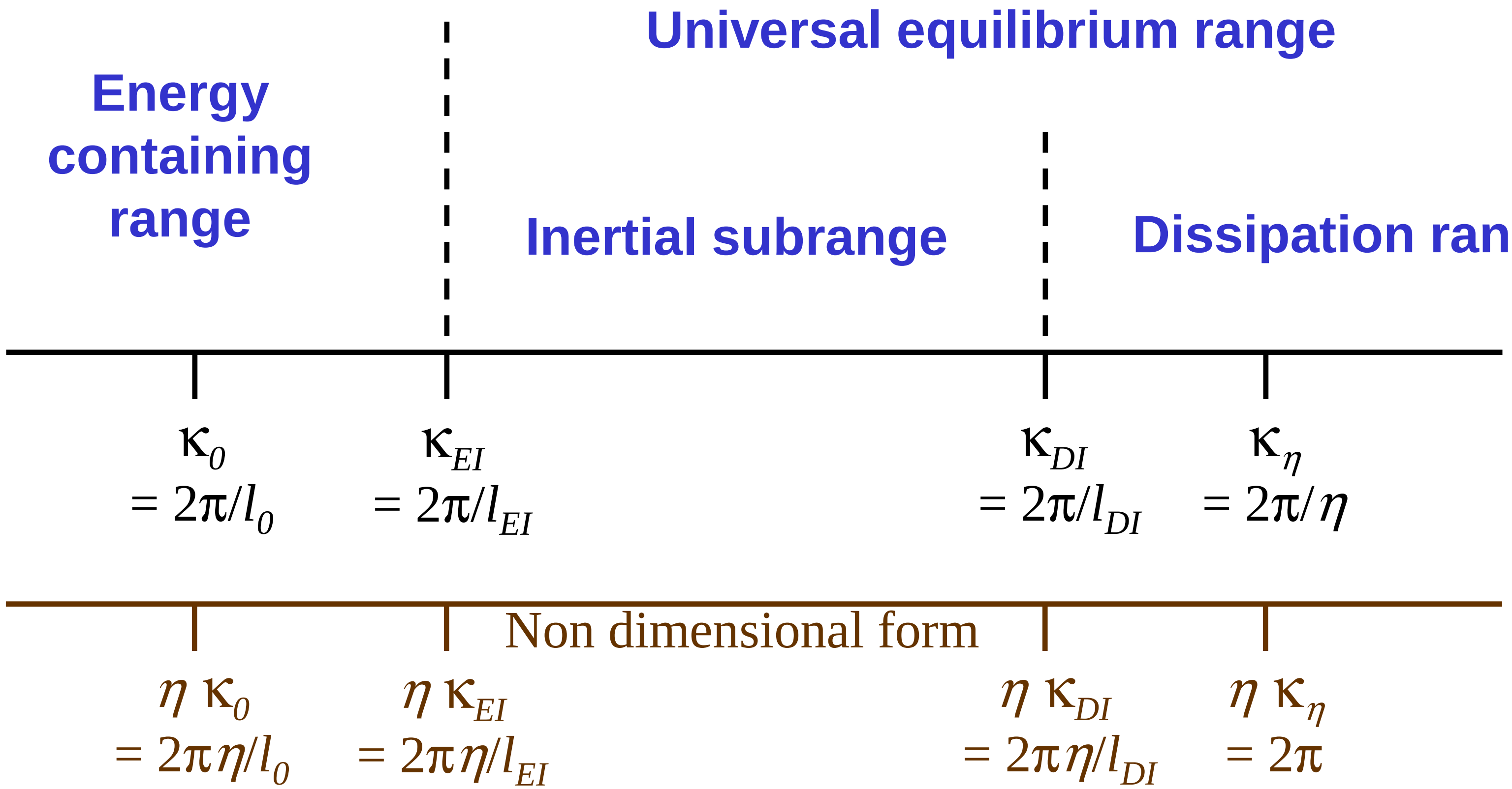
Energy transfer rate

- The rate at which energy is transferred from the larger scales to the smaller scales is T(l).
- Under the equilibrium conditions in the inertial subrange this is equal to the dissipation rate ε, and is proportional to $u(l)^2/\tau$.



Wavenumbers

- The wavenumber κ is defined as $\kappa = 2\pi/l$.
- The different ranges can be shown as a function of wavenumber.
- The wavenumber can also be made non-dimensional by multiplying it with the Kolmogorov length scale η to result in the commonly used dimensionless group $(\eta\ \kappa)$.



Energy spectrum

- The turbulent kinetic energy *k* is given by: $k = \frac{1}{2} < u_i u_i > = \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$
- It remains to be determined how the turbulent kinetic energy is distributed among the eddies of different sizes.
- This is usually done by considering the energy spectrum E(κ).
- Here E(κ) is the energy contained in eddies of size *l* and wavenumber κ, defined as κ = 2π/*l*.
- By definition *k* is the integral of E(κ) over all wavenumbers:

$$k = \int_0^\infty E(\kappa) d\kappa$$

- The energy contained in eddies with wavenumbers between κ_A and κ_B is then:

$$k_{(\kappa_A, \kappa_B)} = \int_{\kappa_A}^{\kappa_B} E(\kappa) d\kappa$$

E(κ) in inertial subrange

- We will develop an equation for E(κ) in the inertial subrange.
- According to the second similarity hypothesis E(κ) will solely depend on κ and ε.
- We can then perform the following dimensional analysis:

$$[k] = m^2 s^{-2} \; ; \;\; [\varepsilon] = m^2 s^{-3} \; ; \;\; [\kappa] = m^{-1} \; ;$$

$$[E(\kappa)] = [k]/[\kappa] = m^3 s^{-2}$$

$$Dimensional\ analysis: \;\; [\varepsilon^{2/3} \kappa^{-5/3}] = m^3 s^{-2}$$

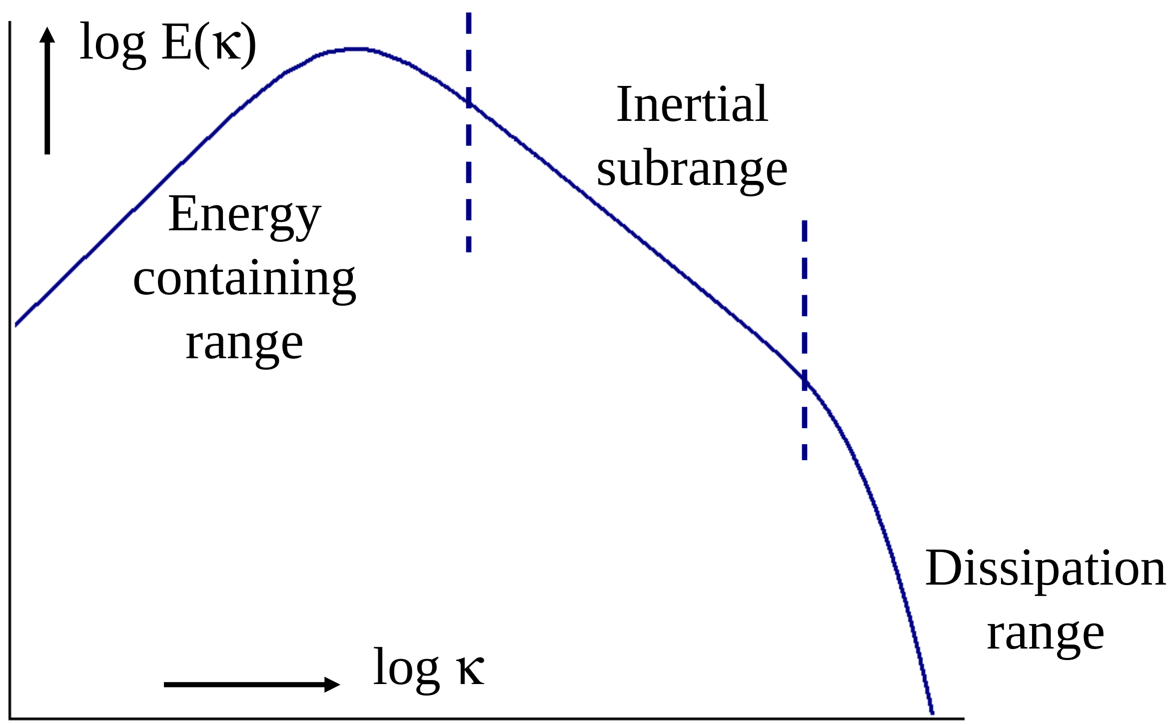
$$\Rightarrow \; E(\kappa) \;\propto\; \varepsilon^{2/3} \kappa^{-5/3}$$

$$\Rightarrow \; E(\kappa) \;=\; C\,\varepsilon^{2/3} \kappa^{-5/3}$$

- The last equation describes the famous Kolmogorov –5/3 spectrum. C is the universal Kolmogorov constant, which experimentally was determined to be C = 1.5.

Full spectrum E(κ)

- Model equations for E(κ) in the production range and dissipation range have been developed. We will not discuss the theory behind them here.
- The full spectrum is given by: $E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3} f_L \; f_\eta$
- Will not discuss f_L and f_η today.



Full spectrum E(κ)

- Model equations for E(κ) in the production range and dissipation range have been developed. We will not discuss the theory behind them here.
- The full spectrum is given by: $E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3} f_L \; f_\eta$
- The production range is governed by f_L (which goes to unity for large ($\kappa\, l_0$):

$$f_L = \left(\frac{\kappa l_0}{\left[(\kappa l_0)^2 + c_L \right]^{1/2}} \right)^{p_0 + 5/3}$$

- The dissipation range is governed by f_η (which goes to unity for small ($\kappa\, \eta$):

$$f_\eta = \exp\{-\beta\{[(\kappa\eta)^4 + c_\eta^4]^{1/4} - c_\eta\}\}$$

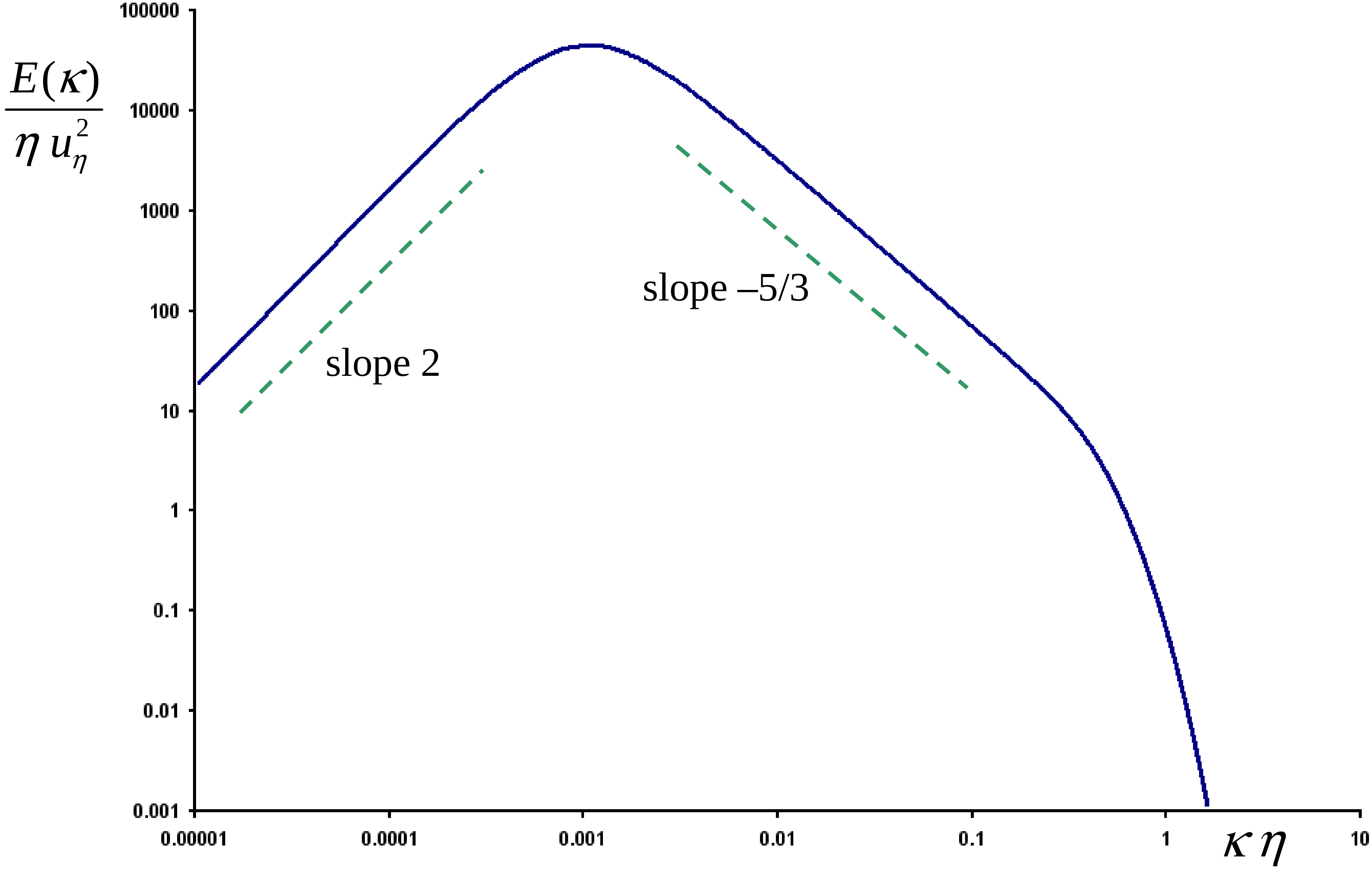
- The model constants were determined experimentally and based on the constraint that E(κ) integrate to *k*. Their values are:

$$c_L \approx 6.78; \quad c_\eta \approx 0.40; \quad C = 1.5; \quad p_0 = 2; \quad \beta = 5.2.$$

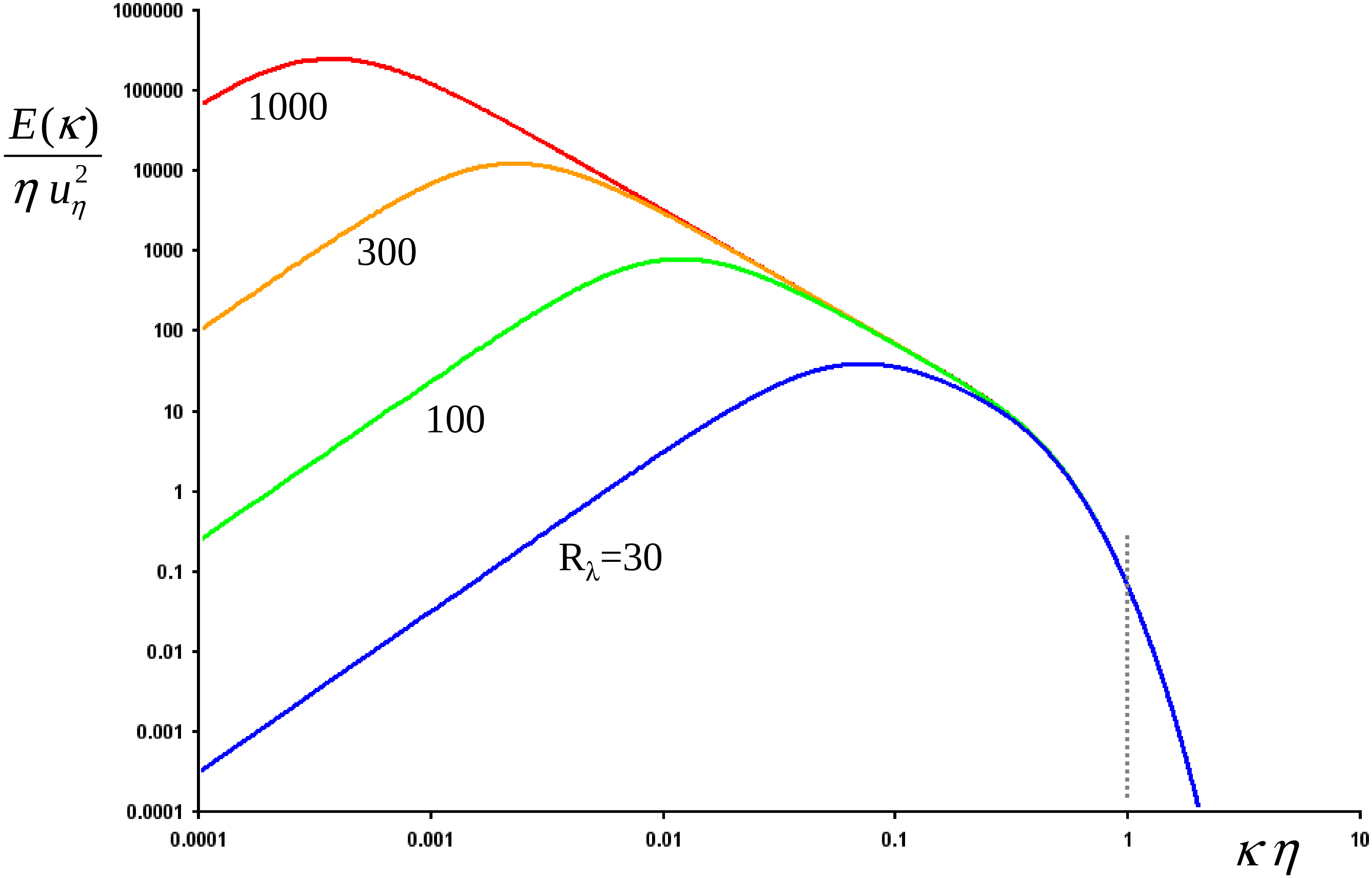
Normalized spectrum

- For given values of ε, ν, and k the full spectrum can now be calculated based on these equations.
- It is, however common to normalize the spectrum in one of two ways: based on the Kolmogorov scales or based on the integral length scale.
- Based on Kolmogorov scale:
  - Measure of length scale becomes ($\eta\ \kappa$).
  - E(κ) is made dimensionless as E(κ)/($\eta\ u_\eta^2$)
- Based on integral scale:
  - Measure of length scale becomes ($l_0\ \kappa$).
  - E(κ) is made dimensionless as E(κ)/($k\ l_0$)
- Instead of having three adjustable parameters (ε,ν,k), the normalized spectrum then has only one adjustable parameter: R_λ.

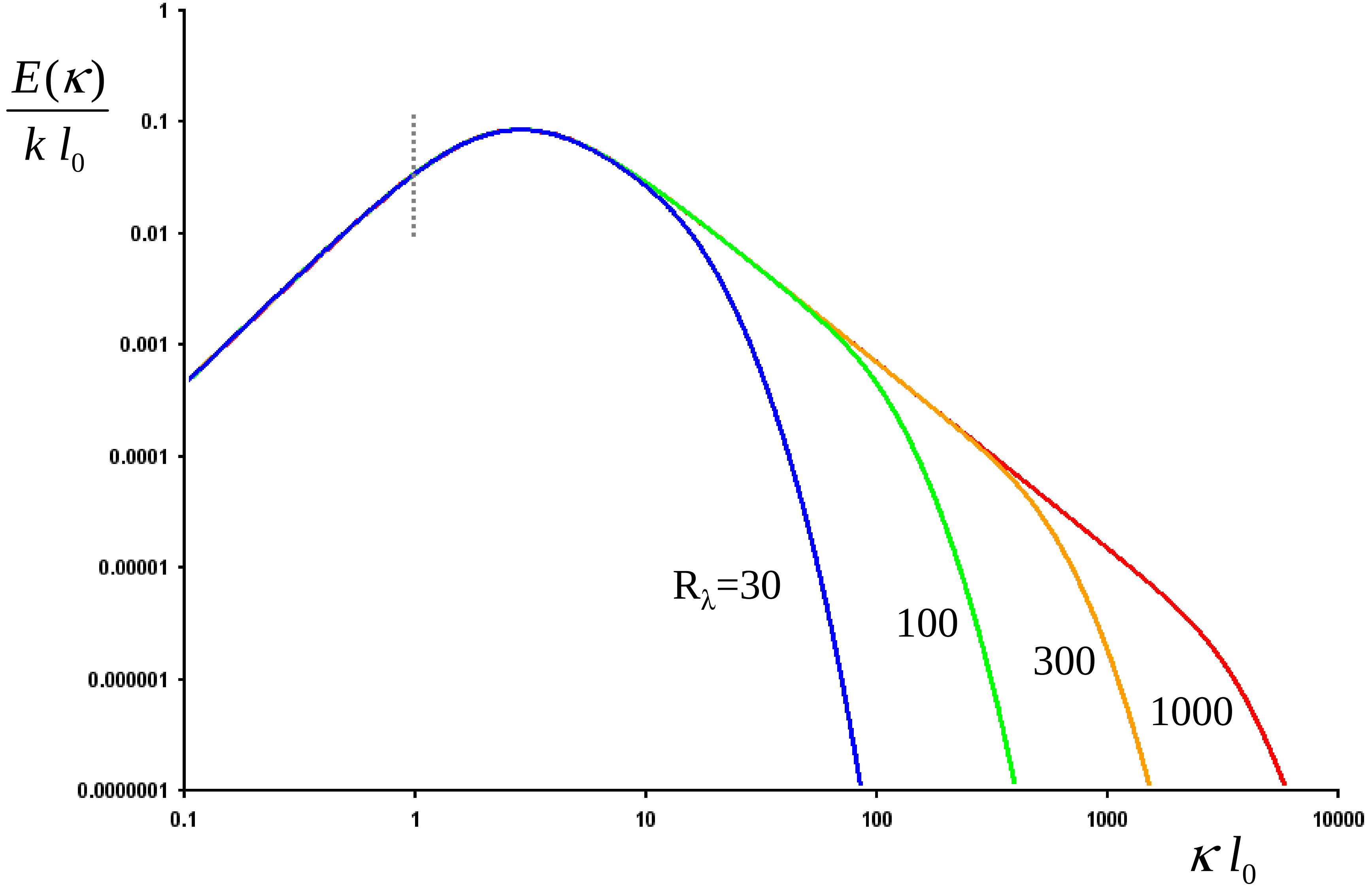
The normalized energy spectrum for $R_\lambda = 500$



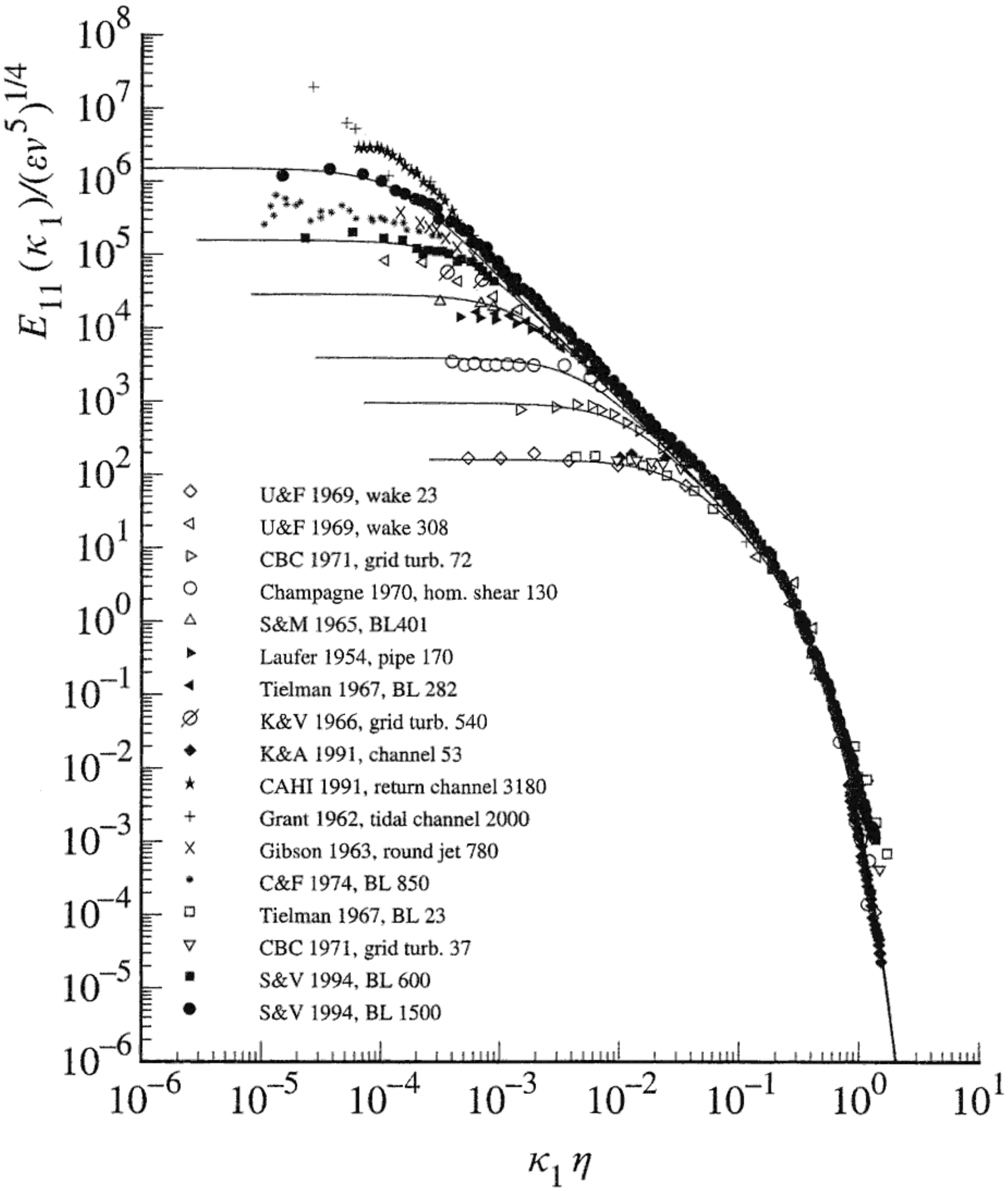
The energy spectrum as a function of R_λ



The energy spectrum as a function of R_λ



Measurements of spectra



The figure shows experimentally measured one dimensional spectra (one velocity component was measured only, as indicated by the “1” and “11” subscripts). The number at the end of the reference denotes the value of R_λ for which the measurements were done. Source: Pope, page 235.

Determination of the spectrum requires simultaneous measurements of all velocity components at multiple points, which is usually not possible. It is common to measure one velocity component at one point over a certain period of time and convert the time signal to a spatial signal using x = Ut with U being the time averaged velocity. This is commonly referred to as Taylor’s hypothesis of frozen turbulence. It is only valid for u’/U << 1, which is not always the case. Spectrum measurements remain a challenging field of research.

Energy containing range

- From the energy spectrum, we can derive which length scales contain most of the turbulent kinetic energy in the flow.
- The derivation will not be reproduced here.
- The conclusion is that most of the energy (~80%) is contained in eddies of length scale $l_{EI} = l_0/6 < l < 6l_0$.

The dissipation rate spectrum

- We now know which eddies contain most of the energy. The question remains, which eddies exactly dissipate the energy.
- This question can be answered by constructing a dissipation rate spectrum D(κ). The integral of D(κ) over the full wavelength range is by definition the energy dissipation rate ε:

$$\varepsilon = \int_0^\infty D(\kappa) d\kappa$$

- Furthermore, with ε being defined as the multiple of the kinematic viscosity and squared velocity gradients (of order $\nu(du/dx)^2 \sim \nu k/l^2 \sim \nu k\kappa^2 \sim \nu\kappa^2 E(\kappa)$) we can then deduce:

$$D(\kappa) = 2\nu\kappa^2 E(\kappa)$$

Dissipation rate spectrum - continued

- This then leads to the following:

$$\varepsilon = \int_0^\infty D(\kappa)d\kappa = \int_0^\infty 2\nu\kappa^2 E(\kappa)d\kappa$$

$$\varepsilon(0,\kappa) = \int_0^\kappa D(\kappa)d\kappa$$

- Here $\varepsilon(0,\kappa)$ is the cumulative dissipation; the energy dissipated by eddies with a wavelength between 0 and κ.
- The unit of $D(\kappa)$ is m^3/s^3 and it can thus be normalized with a velocity scale cubed, typically the Kolmogorov velocity scale.
- Just as the normalized $E(\kappa)$ only depended on R_λ, so does the normalized $D(\kappa)$ depend only on R_λ.

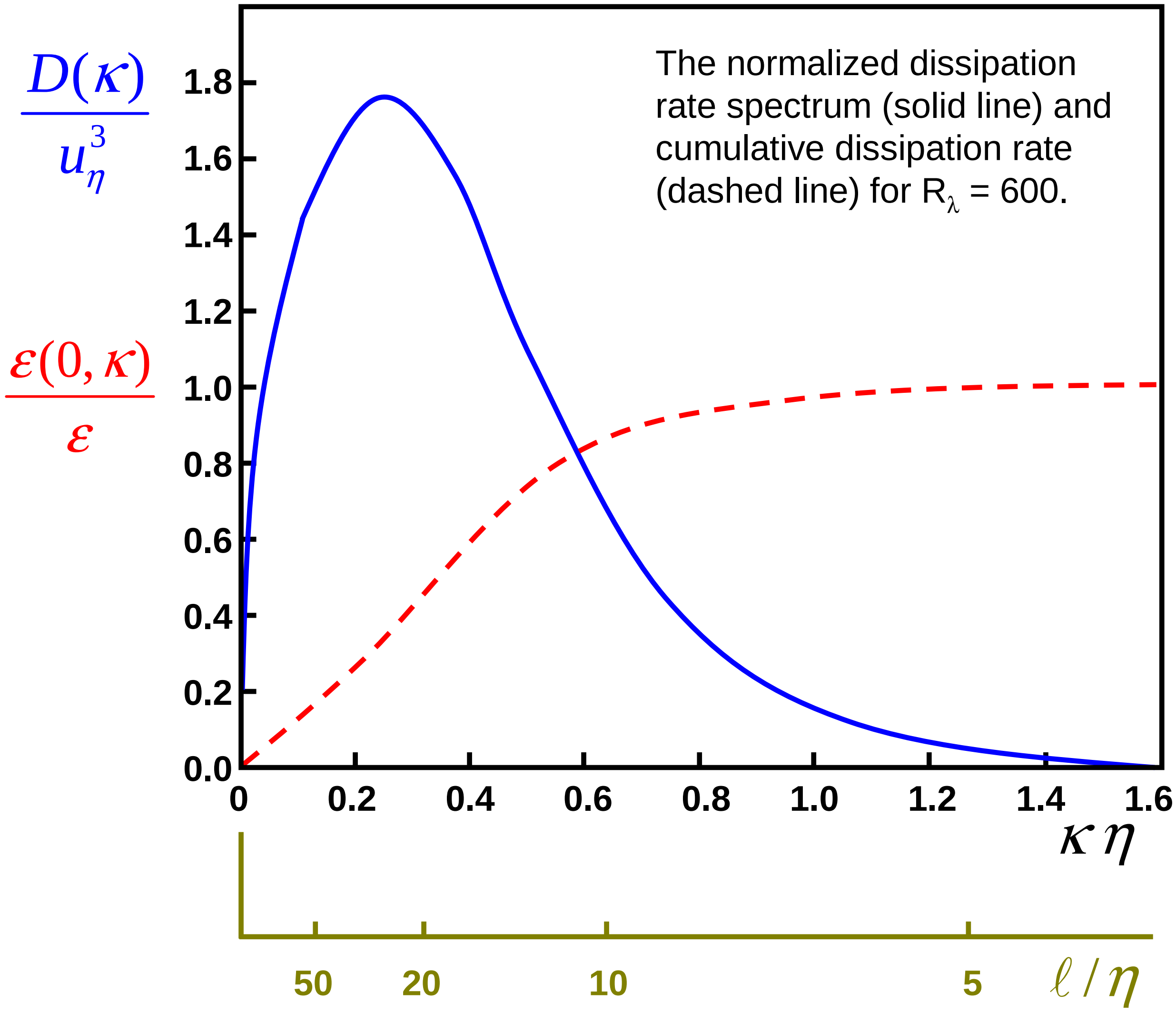
The dissipation rate spectrum

- We now know which eddies contain most of the energy. The question remains, which eddies exactly dissipate the energy.
- This question can be answered by constructing a dissipation rate spectrum D(κ). The integral of D(κ) over the full wavelength range is by definition the energy dissipation rate ε:

$$\varepsilon = \int_0^\infty D(\kappa) d\kappa$$

- The unit of D(κ) is m^3/s^3 and it can thus be normalized with a velocity scale cubed, typically the Kolmogorov velocity scale.
- Just as the normalized E(κ) only depended on R_λ, so does the normalized D(κ) depend only on R_λ.

Dissipation rate spectrum



Dissipation range

- The dissipation rate spectrum can be integrated to show that most of the dissipation (~90%) occurs in eddies of length scales $l_{DI}/\eta = 60 > l/\eta > 8$.
- This means that most of the dissipation occurs at scales that are larger than the Kolmogorov scale η. The Kolmogorov scale should be interpreted as a measure of the smallest eddies that are present in a turbulent flow at high Reynolds numbers.
- How long does it take for a large scale eddy to break up and be dissipated? The spectra can be further analyzed to show that eddies spend about 90% of their total lifetime $\tau = k/\varepsilon$ in the production range, and that once eddies enter the inertial subrange it takes only about $\tau/10$ before the energy is being dissipated. This time $\tau/10$ is also referred to as the cascade timescale.

Intermittency

- Neither *k* nor ε are constant in time or space.
- Within a turbulent flow field, *k* and ε may vary widely in space, sometimes by orders of magnitude.
- Also, at a given point in space the instantaneous values of ε may vary in time. This is called intermittency. The peak values of ε relative to the mean tend to increase with Reynolds number. Peak values may be of the order of 15 times the average ε in laboratory scale flows and 50 times the average in atmospheric flows.

Summary – Reynolds numbers

- The following Reynolds numbers have been defined:
  - Flow Reynolds number: $\mathrm{Re} = UL/\nu$
  - Turbulence Reynolds number: $\mathrm{Re}_L = k^2/\varepsilon\nu$
  - Taylor Reynolds number: $R_\lambda = u'\lambda/\nu$
  - Kolmogorov Reynolds number: $\mathrm{Re}_\eta = \eta u_\eta/\nu = 1$
- The flow Reynolds number is on the order of one to ten times the turbulence Reynolds number.
- The turbulence and Taylor Reynolds numbers can be related as follows: $R_\lambda = \left(\frac{20}{3}\mathrm{Re}_L\right)^{1/2}$

Summary – length scales

- The integral length scale is a measure of the large scale eddies in the production range: $l_0 \propto k^{3/2}/\varepsilon$

  The proportionality constant is of the order one.
- The Taylor microscale is a measure of the size of the eddies in the inertial subrange: $\lambda \approx (10\nu k/\varepsilon)^{1/2}$
- The Kolmogorov microscale is the size of the smallest eddies present in the flow: $\eta = (\nu^3/\varepsilon)^{1/4}$
- The length scales can be related as follows:

$$\lambda / l_0 = \sqrt{10}\,\mathrm{Re}_L^{-1/2}$$

$$\eta / l_0 = \mathrm{Re}_L^{-3/4}$$

$$\lambda / \eta = \sqrt{10}\,\mathrm{Re}_L^{1/4}$$

$$\lambda = \sqrt{10}\,\eta^{2/3}\,l_0^{1/3}$$

Validity of Kolmogorov's theory

- Kolmogorov's theory is an asymptotic theory: it has been shown to work well in the limit of very high Reynolds numbers.
- The exact shape of the normalized spectra may deviate from Kolmogorov's model spectra for intermediate Reynolds numbers. E.g. for many laboratory scale flows which have Reynolds numbers on the order of 10,000 with $R_\lambda \sim 250$, the exponent of $E(\kappa) \sim \kappa^{-p}$ in the inertial subrange is often measured to be $p \sim 1.5$ instead of 5/3 (~1.67).
- Kolmogorov's theory assumes that the energy cascade is one way: from large eddies to small eddies. Experimental studies have shown that energy is also transferred from smaller scales to larger scales (a process called backscatter), albeit at a much lower rate and the dominant energy transfer is indeed from large to small.
- The theory assumes that turbulence at high Reynolds numbers is completely random. In practice, large scale coherent structures may form.
- Research into the fundamental aspects of turbulence continues, both experimentally and by means of large computer simulations using DNS (direct numerical simulation); and the theory continues to be refined.

Sources

- Pope, Stephen B. “Turbulent Flows.” Cambridge University Press 2000.
- Tennekes H., Lumley J.L. “A First Course in Turbulence.” The MIT Press 1972.