Mathematics Review

Applied Computational Fluid Dynamics

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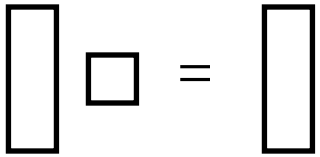
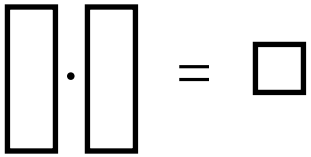
Vector Notation - 1

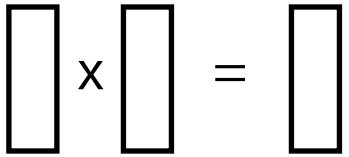
Multiplication: $\vec{X}\ f = \vec{Y}$ $\quad y_i = fx_i$



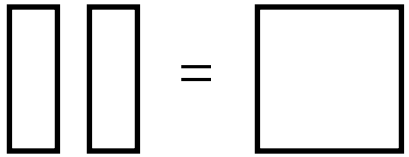
Scalar ("inner") product: $\vec{X} \cdot \vec{Y} = x_i y_i$



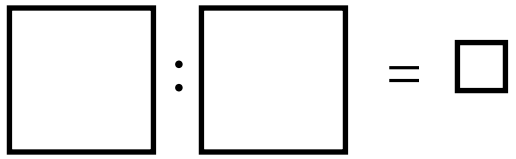
Vector ("outer") product: $\vec{X} \times \vec{Y} = \vec{Z}$



Dyadic ("tensor") product: $\vec{X}\,\vec{Y} = \vec{\vec{A}}$ $\quad a_{ij} = x_i y_j$

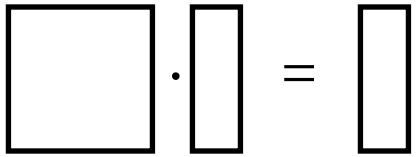


Double dot ("inner") product: $\vec{\vec{A}} : \vec{\vec{B}} = a_{ij} b_{ji}$

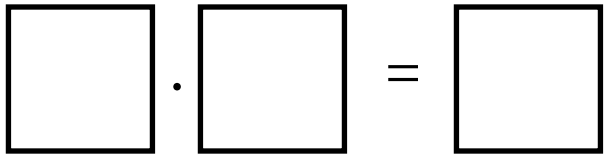


Vector Notation - 2

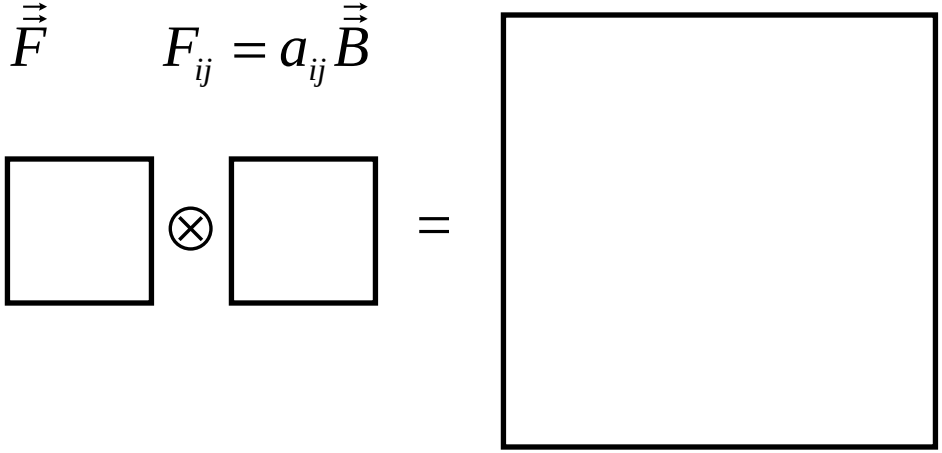
Scalar product: $\vec{\vec{A}} \cdot \vec{X} = \vec{Y}$ $\quad y_i = a_{ik} x_k$



Scalar product: $\vec{\vec{A}} \cdot \vec{\vec{B}} = \vec{\vec{C}}$ $\quad c_{ij} = a_{ik} b_{kj}$



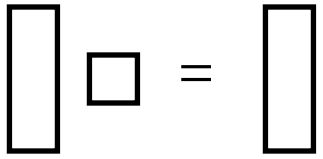
Kronecker product: $\vec{\vec{A}} \otimes \vec{\vec{B}} = \vec{\vec{F}}$ $\quad F_{ij} = a_{ij} \vec{\vec{B}}$



Gradients and divergence

- The gradient of a scalar f is a vector:

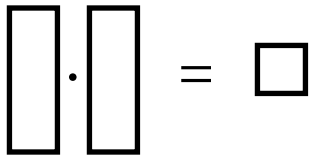
$$grad\ f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$



Similarly, the gradient of a vector is a second order tensor, and the gradient of a second order tensor is a third order tensor.

- The divergence of a vector is a scalar:

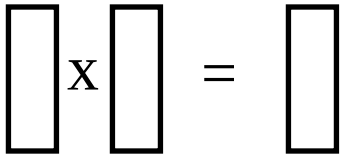
$$div\,\mathbf{A} = \nabla.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$



Curl

- The curl (“rotation”) of a vector **v**(u,v,w) is another vector:

$$curl\ \mathbf{v} = rot\ \mathbf{v} = \nabla \times \mathbf{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$



Definitions - rectangular coordinates

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\nabla . \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{i} + \nabla^2 A_y \mathbf{j} + \nabla^2 A_z \mathbf{k}$$

Identities

$$\nabla(f\,g) = f\,\nabla g + g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$$

$$\nabla\cdot(f\,\mathbf{A}) = (\nabla f)\cdot\mathbf{A} + f\,(\nabla\cdot\mathbf{A})$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(f\mathbf{A}) = (\nabla f)\times\mathbf{A} + f\,(\nabla\times\mathbf{A})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B}$$

Identities

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$(\mathbf{A} \cdot \nabla)\mathbf{B} = \left[A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right] \mathbf{i}$$
$$+ \left[A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z} \right] \mathbf{j}$$
$$+ \left[A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z} \right] \mathbf{k}$$

$$\int_\tau \nabla f \, d\tau = \int_S f \, \mathbf{da}$$

$$\int_\tau (\nabla \times \mathbf{A}) \, d\tau = -\int_S \mathbf{A} \times \mathbf{da}$$

where S is the surface which bounds volume τ

Differentiation rules

$\varphi = \varphi(\boldsymbol{r})$ en $\psi = \psi(\boldsymbol{r})$ *are differentiable scalar functions*
$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r})$ en $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{r})$ *are differentiable vector functions*

$\text{grad}\ \varphi = \nabla\varphi$; $\text{div}\ \boldsymbol{v} = \nabla\cdot\boldsymbol{v}$; $\text{rot}\ \boldsymbol{v} = \nabla \times \boldsymbol{v}$

$$
\begin{aligned}
&\text{grad}\,(\varphi + \psi) = \text{grad}\,\varphi + \text{grad}\,\psi \\
&\text{div}\,(\boldsymbol{v} + \text{w}) \quad = \text{div}\,\boldsymbol{v} + \text{div}\,\boldsymbol{w} \\
&\text{rot}\,(\boldsymbol{v} + \text{w}) \quad = \text{rot}\,\boldsymbol{v} + \text{rot}\,\boldsymbol{w} \\
&\text{grad}\,(\varphi\psi) \quad = \psi\ \text{grad}\,\varphi + \varphi\ \text{grad}\,\psi \\
&\text{div}\,(\varphi\boldsymbol{v}) \quad = v\cdot\text{grad}\ \varphi + \varphi\,\text{div}\ \boldsymbol{v} \\
&\text{rot}\,(\varphi\boldsymbol{v}) \quad = \text{grad}\,\varphi \times \boldsymbol{v} + \varphi\,\text{rot}\ v \\
&\text{div}\,(\boldsymbol{v} \times w) \quad = \boldsymbol{w}\cdot\text{rot}\ \boldsymbol{v} - \boldsymbol{v}\cdot\,\text{rot}\,\boldsymbol{w} \\
&\text{rot}\,(\boldsymbol{v} \times w) \quad = (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} - w\ \text{div}\ \boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} + \boldsymbol{v}\ \text{div}\,\boldsymbol{w} \\
&\text{grad}\,(\boldsymbol{v}\cdot\boldsymbol{w}) \quad = \boldsymbol{w} \times \text{rot}\,\boldsymbol{v} + (\boldsymbol{w}\cdot\nabla)v + \boldsymbol{v} \times \text{rot}\,\boldsymbol{w} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} \\
&\text{div grad}\,\varphi \quad = \nabla^2\varphi (\varphi = \varphi(\boldsymbol{r}) \\
&\qquad \nabla^2 \quad = \nabla\cdot\nabla = \partial_x^2 + \partial_y^2 + \partial_z^2\ \text{(Laplace operator)} \\
&\text{rot grad}\,\varphi \quad = \mathbf{0} \\
&\text{div rot}\,\boldsymbol{v} \quad = 0 \qquad\qquad (v = v(\boldsymbol{r})\ \textit{twice} \\
&\text{rot rot}\ \boldsymbol{v} \quad = \text{grad div}\ v - \nabla^2\boldsymbol{v} \qquad\qquad \textit{differentiable})
\end{aligned}
$$

Integral theorems

Gauss' divergence theorem: $\iint_S \mathbf{A} \cdot d\mathbf{s} = \iiint_\tau \nabla \cdot \mathbf{A}\, d\tau$

where S is the surface which bounds volume τ

Stokes' theorem: $\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$

where C is the closed curve which bounds the open surface S (*i.e. S may be a* $3-D$ *surface but does not bound a volume*)

Euclidian Norm

- Various definitions exist for the norm of vectors and matrices. The most well known is the Euclidian norm.

- The Euclidian norm ||**V**|| of a vector **V** is:

$$\|\mathbf{V}\| = \sqrt{\sum_i v_i^2}$$

- The Euclidian norm ||**A**|| of a matrix **A** is:

$$\|\mathbf{A}\| = \sqrt{\sum_{i,j} a_{ij}^2}$$

- Other norms are the spectral norm, which is the maximum of the individual elements, or the Hölder norm, which is similar to the Euclidian norm, but uses exponents p and 1/p instead of 2 and 1/2, with p a number larger or equal to 1.

Matrices - Miscellaneous

- The determinant of a matrix **A** with elements a_{ij} and i=3 rows and j=3 columns:

$$\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det \mathbf{A} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

- A diagonal matrix is a matrix where all elements are zero except a_{11}, a_{22}, and a_{23}. For a tri-diagonal matrix also the diagonals above and below the main diagonal are non-zero, while all other elements are zero.
- Triangular decomposition is expressing **A** as the product **LU** with **L** a lower-triangular matrix (elements above diagonal are 0) and **U** an upper triangular matrix.
- The transpose $\mathbf{A}^T$ has elements $a'_{ij}=a_{ji}$. A matrix is symmetric if $\mathbf{A}^T = \mathbf{A}$.
- A sparse matrix is a matrix where the vast majority of elements is zero and only few elements are non-zero.

Matrix invariants - 1

- An invariant is a scalar property of a matrix that is independent of the coordinate system in which the matrix is written.
- The first invariant I_1 of a matrix **A** is the trace *tr* **A**. This is simply the sum of the diagonal components: $I_1 = tr\ \mathbf{A} = a_{11} + a_{22} + a_{33}$
- The second invariant is: $I_2 = \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix}$
- The third invariant is the determinant of the matrix: $I_3 = \det \mathbf{A}$.
- The three invariants are the simplest possible linear, quadratic, and cubic combinations of the eigenvalues that do not depend on their ordering.

Matrix invariants - 2

- Using the previous definitions we can form infinitely many other variants, e.g:

$$I_1^2 = (a_{ii})^2$$

$$I_1^2 - 2I_2 = a_{ik}a_{ik}$$

- In CFD literature, one will sometimes encounter the following alternative definitions of the second invariant (which are derived from the previous definitions):
  - For symmetric matrices:

    $I_2 = (1/2)*[(\text{tr } \mathbf{A})^2 - \text{tr } \mathbf{A}^2] = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11}$

    or

    $I_2 = (1/6) * [(a_{11}-a_{22})^2 + (a_{22}-a_{33})^2 + (a_{33}-a_{11})^2] + a_{12}^2 + a_{23}^2 + a_{31}^2$

  - The Euclidian norm: $I_2 = \|\mathbf{A}\| = \sqrt{\sum_{i,j} a_{ij}^2}$

Gauss-Seidel method

- The Gauss-Seidel method is a numerical method to solve the following set of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1N}x_N = C_1$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 + ... + a_{NN}x_N = C_N$$

- We first make an initial guess for x_1:

$$x_1^1 = \frac{C_1}{a_{11}}$$

- The superscript 1 denotes the 1st iteration.
- Next, using x_1^1:

$$x_2^1 = \frac{C_2}{a_{22}} - \frac{1}{a_{22}}(a_{21}x_1^1)$$

Gauss-Seidel method - continued

- Next, using x_1^1 and x_2^0:

$$x_3^1 = \frac{C_3}{a_{33}} - \frac{1}{a_{33}}(a_{31}x_1^1 + a_{32}x_2^1)$$

- And continue, until:

$$x_N^1 = \frac{C_N}{a_{NN}} - \frac{1}{a_{NN}}\sum_1^{n-1} a_{Ni}x_i^1$$

- For all consecutive iterations we solve for x_1^2, using x_2^1 … x_N^1, and next for x_2^2 using x_1^2, x_3^1 … x_N^1, etc.
- We repeat this process until convergence, i.e. until:

$$(x_i^k - x_i^{k-1}) < \delta$$

with δ a specified small value.

Gauss-Seidel method - continued

- It is possible to improve the speed at which this system of equations is solved by applying overrelaxation, or improve the stability if the system does not converge by applying underrelaxation.
- Say at iteration k the value of x_i equals x_i^k. If applying the Gauss-Seidel method, the value for iteration k+1 would be x_i^{k+1}, then, instead of using x_i^{k+1}, we consider this to be a *predictor*.
- We then calculate a *corrector* as follows:

$$corrector = R(x_i^{k+1} - x_i^k)$$

- Here R is the relaxation factor (R>0). If R<1 we use underrelaxation and if R>1 we use overrelaxation.
- Next we recalculate x_i^{k+1}as follows:

$$x_i^{k+1} = x_i^k + corrector$$

Gauss elimination

- Consider the same set of algebraic equations shown in the Gauss-Seidel discussion. Consider the matrix **A**:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The heart of the algorithm is the technique for eliminating all the elements below the diagonal, i.e. to replace them with zeros, to create an upper triangular matrix:

$$\mathbf{U} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Gauss elimination - continued

- This is done by multiplying the first row by a_{21}/a_{11} and subtracting it from the second row. Note that C_2 then becomes $C_2-C_1a_{21}/a_{11}$.
- The other elements a_{31} through a_{n1} are treated similarly. Now all elements in the first column below a_{11} are 0.
- This process is then repeated for all columns.
- This process is called forward elimination.
- Once the upper diagonal matrix has been created, the last equation only contains one variable x_n, which is readily calculated as $x_n=C_n/a_{nn}$.
- This value can then be substituted in equation n-1 to calculate x_{n-1} and this process can be repeated to calculate all variables x_i. This is called backsubstitution.
- The number of operations required for this method increases proportional to n^3. For large matrices this can be a computationally expensive method.

Tridiagonal matrix algorithm (TDMA)

- TDMA is an algorithm similar to Gauss elimination for tridiagonal matrices, i.e. matrices for which only the main diagonal and the diagonals immediately above and below it are non-zero.
- This system can be written as:

$$a_{i,i-1}x_{i-1} + a_{i,i}x_i + a_{i,i+1}x_{i+1} = C_i$$

- Only one element needs to be replaced by a zero on each row to create an upper diagonal matrix.
- When the algorithm reaches the ith row, only $a_{i,i}$ and C_i need to be modified:

$$a_{i,i} = a_{i,i} - \frac{a_{i,i-1}a_{i-1,i+1}}{a_{i-1,i}} \qquad C_i = C_i - \frac{a_{i,i-1}C_{i-1}}{a_{i-1,i}}$$

- Backsubstitution is then used to calculate all x_i.
- The computational effort scales with n and this is an efficient method to solve this set of equations.

Differential equations

- Ordinary differential equation (ODE): an equation which, other than the one independent variable x and the dependent variable y, also contains derivatives from y to x. General form:

  $$F(x,y,y',y'' \ldots y^{(n)}) = 0$$

  The order of the equation is determined by the order n of the highest order derivative.

- A partial differential equation (PDE) has two or more independent variables. A PDE with two independent variables has the following form:

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \ldots \right) = 0$$

  with $z=z(x,y)$. The order is again determined by the order of the highest order partial derivative in the equation. Methods such as “Laplace transformations” or “variable separation” can sometimes be used to express PDEs as sets of ODEs. These will not be discussed here.

Classification of partial differential equations

- A general partial differential equation in coordinates x and y:

$$a\frac{\partial^2\phi}{\partial x^2}+b\frac{\partial^2\phi}{\partial x\partial y}+c\frac{\partial^2\phi}{\partial y^2}+d\frac{\partial\phi}{\partial x}+e\frac{\partial\phi}{\partial y}+f\phi+g=0$$

- Characterization depends on the roots of the higher order (here second order) terms:
  - $(b^2-4ac) > 0$ then the equation is called hyperbolic.
  - $(b^2-4ac) = 0$ then the equation is called parabolic.
  - $(b^2-4ac) < 0$ then the equation is called elliptic.
- Note: if a, b, and c themselves depend on x and y, the equations may be of different type, depending on the location in x-y space. In that case the equations are of *mixed* type.

Origin of the terms

- The origin of the terms “elliptic,” “parabolic,” or “hyperbolic used to label these equations is simply a direct analogy with the case for conic sections.
- The general equation for a conic section from analytic geometry is:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where if

- $(b^2-4ac) > 0$ the conic is a hyperbola.
- $(b^2-4ac) = 0$ the conic is a parabola.
- $(b^2-4ac) < 0$ the conic is an ellipse.

Numerical integration methods

- Ordinary differential equation:

$$\frac{d\phi(t)}{dt} = f(t, \phi(t)); \quad \phi(t_0) = \phi^0$$

- Here f is a known function. Φ^0 is the initial point. The basic problem is how to calculate the solution a short time Δt after the initial point at time $t_1 = t_0 + \Delta t$. This process can then be repeated to calculate the solution at t_2, etc.
- The simplest method is to calculate the solution at t_1 by adding $f(t_0, \Phi^0)\,\Delta t$ to Φ^0. This is called the *explicit* or *forward* Euler method, generally expressed as:

$$\phi(t_{n+1}) = \phi^{n+1} = \phi^n + f(t_n, \phi^n)\Delta t$$

Numerical integration methods

- Another method is the trapezoid rule which forms the basis of a popular method to solve differential equations called the Crank-Nicolson method:

$$\phi^{n+1} = \phi^n + \frac{1}{2}\left[f(t_n, \phi^n) + f(t_{n+1}, \phi^{n+1}) \right] \Delta t$$

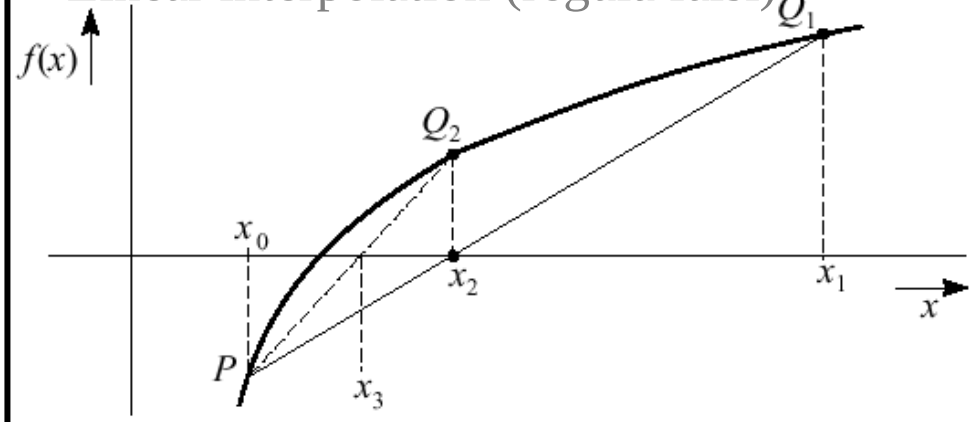
- Methods using points between t_n and t_{n+1} are called Runge-Kutta methods, which come in various forms. The simplest one is second-order Runge-Kutta:

$$\phi^*_{n+1/2} = \phi^n + \frac{\Delta t}{2} f(t_n, \phi^n)$$

$$\phi^{n+1} = \phi^n + \Delta t \, f(t_{n+1/2}, \phi^*_{n+1/2})$$

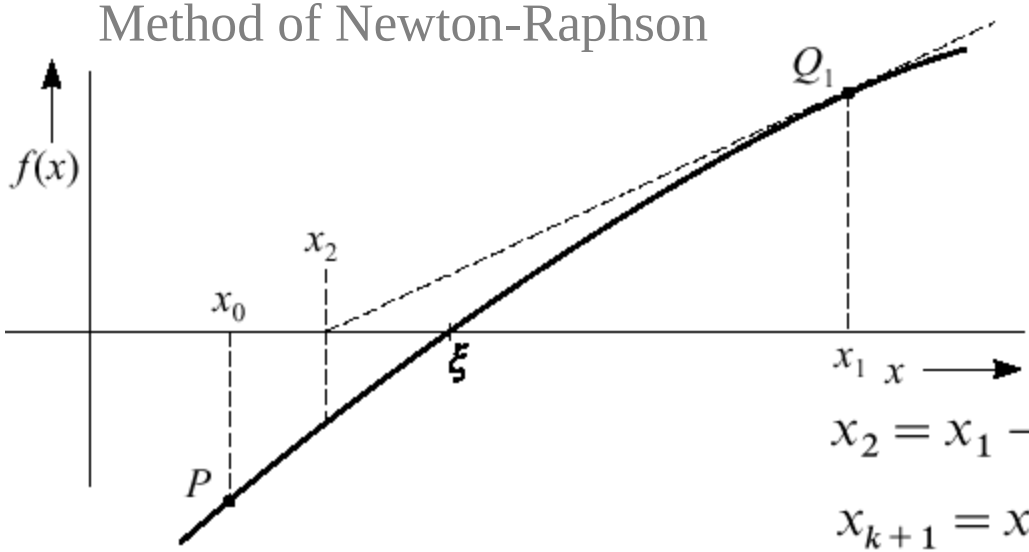
Numerically estimating zero-crossings

Linear interpolation (regula falsi)



$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$ (use recursively)

Method of Newton-Raphson



$$x_2 = x_1 - f(x_1)/f'(x_1)$$

$$x_{k+1} = x_k - f(x_k)/f'(x_k)$$

Jacobian

- The general definition of the Jacobian for n functions of n variables is the following set of partial derivatives:

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

- In CFD, the shear stress tensor $S_{ij} = \partial U_i / \partial x_j$ is also called "the Jacobian."

Jacobian - continued

- The Jacobian can be used to calculate derivatives from a function in one coordinate sytem from the derivatives of that same function in another coordinate system.
- Equations u=f(x,y), v=g(x,y), then x and y can be determined as functions of u and v (possessing first partial derivatives) as follows:

$$u = f(x,y); \quad f_x = \partial f / \partial y; \quad f_y = \partial f / \partial x$$

$$v = g(x,y); \quad g_x = \partial g / \partial x; \quad g_y = \partial g / \partial y$$

$$\frac{\partial x}{\partial u} = \frac{g_y}{\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}} \qquad \frac{\partial y}{\partial u} = \frac{-g_x}{\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}}$$

- With similar functions for x_v and y_v.
- The determinants in the denominators are examples of the use of Jacobians.

Eigenvalues

- If an equation with an adjustable parameter has non-trivial solutions only for specific values of that parameter, those values are called the eigenvalues and the corresponding function the eigenfunction.
- If a differential equation with an adjustable parameter only has a solution for certain values of that parameter, those values are called the eigenvalues of the differential equation.
- For an *n*x*n* matrix **A**, for the equation $\mathbf{Az} = \lambda\mathbf{z}$, then z is an eigenvector and λ is an eigenvalue.
    - The eigenvalues are the *n* roots of the characteristic equation

      $$\det(\lambda\mathbf{I}-\mathbf{A}) = \lambda^n+p_1\lambda^{n-1}+\ldots+p_n = 0$$

    - $(\lambda\mathbf{I}-\mathbf{A})$ is the characteristic matrix of **A**.
    - The polynomial is called the characteristic polynomial of **A**.
    - The product of all the eigenvalues of **A** is equal to det **A**.
    - The sum of all the eigenvalues is equal to tr **A**.
    - The matrix is singular if at least one eigenvalue is zero.

Taylor series

- Let $f(x)$ have continuous derivatives up to the $(n+1)^{st}$ order in some interval containing the point a. Then:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n + ...$$

Error function

- The error function is defined as:

$$Error\ \ function: \quad erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

$$Complementary\ error\ \ function: \quad erfc(z) = 1 - erf(z)$$

- It is the integral of the Gaussian ("normal") distribution. It is usually calculated from series expansions.
- Properties are:

$$erf(0) = 0$$

$$erf(\infty) = 1$$

$$erf(z) = -erf(-z)$$

$$\frac{d\, erf(z)}{dz} = \frac{2}{\sqrt{\pi}} \exp\left(-z^2\right)$$

Permutation symbol

- The permutation symbol e_{kmn} resembles a third-order tensor.
- If the number of transpositions required to bring *k*, *m*, and *n* in the form 1, 2, 3 is even then e_{kmn}=1.
- If the number of transpositions required to bring *k*, *m*, and *n* in the form 1, 2, 3 is odd then e_{kmn}=-1.
- Otherwise e_{kmn}=0.
- Thus:

$$e_{123} = e_{231} = e_{312} = 1$$

$$e_{132} = e_{321} = e_{213} = -1$$

all other elements are zero

- Instead of e_{kmn} the permutation symbol is also often written as ε_{kmn}

Correlation functions

- Continuous signals.
  - Let *x(t)* and *y(t)* be two signals. Then the correlation function $\Phi_{xy}(t)$ is defined as:

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)\, y(\tau)\, d\tau$$

  - The function $\Phi_{xx}(t)$ is usually referred to as the autocorrelation function of the signal, while the function $\Phi_{xy}(t)$ is usually called the cross-correlation function.
- Discrete time series.
  - Let *x[n]* and *y[n]* be two real-valued discrete-time signals. The autocorrelation function $\Phi_{xx}[n]$ of *x[n]* is defined as:

$$\phi_{xx}[n] = \sum_{m=-\infty}^{+\infty} x[m+n]\, x[m]$$

  - The cross-correlation function is defined as:

$$\phi_{xy}[n] = \sum_{m=-\infty}^{+\infty} x[m+n]\, y[m]$$

Fourier transforms

- Fourier transforms are used to decompose a signal into a sum of complex exponentials (i.e. sinusoidal signals) at different frequencies.
- The general form is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}d\omega$$

- X(ω) is the Fourier transform of x(t). It is also called the spectrum because it shows the amplitude ("energy") associated with each frequency ω present in the signal. X(ω) is a complex function with real and imaginary parts. The magnitude |X(ω)| is also called the power spectrum.
- Slightly different forms exist for continuous, discrete, periodic, and aperiodic signals.