

OVERVIEW BALANCE EQUATIONS ISOTHERMAL SINGLE-PHASE FLOW AND  
TURBULENCE MODELS

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OVERVIEW EQUATIONS FOR TIME AVERAGED INCOMPRESSIBLE STEADY STATE FLOW

The continuity equation:

$$\vec{\nabla} \cdot \vec{U} = 0$$

The momentum balance:

$$\vec{\nabla} \cdot (\vec{U}\vec{U}) = \vec{\nabla} \cdot (\vec{\pi} + \vec{\pi}_T)$$

The molecular stress tensor:

$$\vec{\pi} = -\frac{P}{\rho} \vec{I} + \nu (\vec{\nabla}\vec{U} + (\vec{\nabla}\vec{U})^T)$$

The Reynolds stress tensor:

$$\vec{\pi}_T = -\overline{uu}$$

k-ε model:

$$\vec{\pi}_T = -\frac{2}{3} k \vec{I} + \nu_t (\vec{\nabla}\vec{U} + (\vec{\nabla}\vec{U})^T)$$

Turbulent viscosity:

$$\nu_t = c_\mu k^2/\varepsilon$$

Model equation for k:

$$\vec{U} \cdot \vec{\nabla} k = \vec{\nabla} \cdot ((\nu + \nu_t/\sigma_k) \vec{\nabla} k) + P_k - \varepsilon$$

$$P_k = -\overline{uu} : (\vec{\nabla}\vec{U})$$

Model equation for ε:

$$\vec{U} \cdot \vec{\nabla} \varepsilon = \vec{\nabla} \cdot ((\nu + \nu_t/\sigma_\varepsilon) \vec{\nabla} \varepsilon) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

Algebraic stress model:

$$\vec{\pi}_T = \frac{2}{3} k \vec{I} \left[ 1 - \frac{(1-c_2) \frac{P_k}{\varepsilon}}{\frac{P_k}{\varepsilon} - (1-c_1)} \right] + \frac{1-c_2}{\frac{P_k}{\varepsilon} - (1-c_1)} \frac{k}{\varepsilon} \left( \vec{P} - \frac{2}{3} P_k \vec{I} \right)$$

$$\vec{P} = -\left( \overline{uu} : (\vec{\nabla}\vec{U}) + (\overline{uu} : (\vec{\nabla}\vec{U}))^T \right)$$

CONTINUITY EQUATION

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\hat{\rho} \hat{U}_i) = 0$$

MOMENTUM BALANCE

$$\frac{\partial}{\partial t} (\hat{\rho} \hat{U}_i) + \frac{\partial}{\partial x_k} (\hat{\rho} \hat{U}_i \hat{U}_k) = -\frac{\partial \hat{P}}{\partial x_i} + \frac{\partial}{\partial x_k} \left( \mu \left( \frac{\partial \hat{U}_i}{\partial x_k} + \frac{\partial \hat{U}_k}{\partial x_i} - \frac{2}{3} \frac{\partial \hat{U}_m}{\partial x_m} \delta_{ik} \right) \right) + \hat{\rho} g_i$$

$$\frac{2}{3} \frac{\partial \hat{U}_m}{\partial x_m} \delta_{ik} = 0 \quad \text{for incompressible flows}$$

MEAN MOTION - CONTINUITY

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho U_i) = 0$$

For incompressible flows:

$$\frac{\partial U_i}{\partial x_i} = 0$$

MEAN MOTION - MOMENTUM

$$\frac{\partial}{\partial t} (\rho U_i) + \frac{\partial}{\partial x_j} (\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_k} \left( \mu \frac{\partial U_k}{\partial x_k} - \mu \frac{2}{3} \frac{\partial U_m}{\partial x_m} \delta_{ik} - \overline{\rho u_i u_k} \right) + \rho g_i$$

For incompressible flows:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_k} \left( \nu \frac{\partial U_k}{\partial x_k} - \overline{u_i u_k} \right) + g_i$$

DEFINITIONS REYNOLDS AVERAGING

$$\hat{U}_i = U_i + u_i$$

$$\overline{\hat{U}_i \hat{U}_j} = U_i U_j + \overline{u_i u_j}$$

$$\overline{\hat{U}_i} = U_i$$

$$\overline{u_i} = 0$$

$$\hat{\Phi} = \Phi + \varphi$$

$$\overline{\hat{\Phi}} = \Phi$$

$$\overline{\varphi} = 0$$

$$\overline{\hat{\Phi} \hat{U}_i} = \Phi U_i + \overline{\varphi u_i}$$

$$\overline{\hat{\rho}} = \rho$$

$$\hat{P} = P + p$$

REYNOLDS STRESSES - ANALYTICAL EQUATION (incompressible flow)

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = - \underbrace{\left( \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} + \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} \right)}_{(A)}$$

$$- \underbrace{\frac{\partial}{\partial x_k} \left( \overline{u_i u_j u_k} - \nu \frac{\partial}{\partial x_k}(\overline{u_i u_j}) + \frac{p}{\rho} (\overline{u_i \delta_{jk}} + \overline{u_j \delta_{ik}}) \right)}_{(B)}$$

$$+ \underbrace{\frac{p}{\rho} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}_{(C)} - 2 \nu \underbrace{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}_{(D)}$$

(A) Rate of generation of  $\overline{u_i u_j}$  by the effects of the mean strain.  
Transfer of energy from the mean flow to turbulence. Symbol  $P_{ij}$ .

(B) "Diffusion"  $D_{ij}$  of  $\overline{u_i u_j}$  in three components:  
(1) "turbulent" diffusion involving triple products.  
(2) "viscous" diffusion written in this way for convenience.  
(3) "pressure" diffusion.

(C) The pressure-strain correlation. Symbol  $\Psi_{ij}$ . Appears by rearrangement of the pressure terms:

$$\overline{u_j \frac{\partial}{\partial x_i} (P+p)} = \overline{u_j \frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i}(\overline{p u_j}) - p \frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_k}(\overline{p u_j}) - p \frac{\partial u_j}{\partial x_i}$$

This decomposition into pressure diffusion plus pressure strain is convenient because the second component has zero trace on contraction. It is not unique.

(D) Viscous destruction of  $\overline{u_i u_j}$ . Symbol  $\epsilon_{ij}$ . Appears by rearrangement of the viscous terms.

$$\overline{\nu u_j \frac{\partial^2}{\partial x_k^2} (U_i + u_i)} + \overline{\nu u_i \frac{\partial^2}{\partial x_k^2} (U_j + u_j)} = \nu \frac{\partial^2}{\partial x_k^2}(\overline{u_i u_j}) - 2\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}$$

Convenient because the first component has the form of viscous gradient diffusion of  $\overline{u_i u_j}$ . The second component has useful properties.

TURBULENT KINETIC ENERGY - ANALYTICAL EQUATION (Incompressible flow)

$$k = \frac{1}{2} \overline{u_i u_i}$$

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \underbrace{- \overline{u_i u_k} \frac{\partial U_i}{\partial x_k}}_{(A)} - \underbrace{\frac{\partial}{\partial x_k} \left( \frac{1}{2} \overline{u_k u_i^2} - \nu \frac{\partial k}{\partial x_k} + \frac{\overline{p}}{\rho} u_k \right)}_{(B)} - \underbrace{\nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}}_{(C)}$$

- (A) Production rate of k by interaction with the mean strain rate. Symbol  $P_k$ .
- (B) "Diffusion" of k: (1) "Turbulent", (2) "Viscous", (3) "Pressure"
- (C) Viscous destruction of k, symbol  $\epsilon$ .

TURBULENT KINETIC ENERGY - MODEL EQUATION (Incompressible flow)

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k} \left( (\nu + \nu_t / \sigma_k) \frac{\partial k}{\partial x_k} \right) + P_k - \epsilon$$

$$P_k = - \overline{u_i u_k} \frac{\partial U_i}{\partial x_k}$$

$$\nu_t = c_\mu k^2 / \epsilon$$

$$c_\mu = 0.09$$

$$\sigma_k = 1.0$$

This model equation is used in both the low Reynolds no. and the high Reynolds no. k- $\epsilon$  eddy viscosity model, as well as in higher order closure models like the ASM model and full Reynolds stress models.

ENERGY DISSIPATION RATE - ANALYTICAL EQUATION

$$\begin{aligned} \epsilon &= \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}} \\ \frac{\partial \epsilon}{\partial t} + U_k \frac{\partial \epsilon}{\partial x_k} &= - \frac{\partial}{\partial x_k} \left( \overline{\nu u_k \frac{\partial u_i}{\partial x_1} \frac{\partial u_i}{\partial x_1}} + 2 \overline{\frac{\nu}{\rho} \frac{\partial p}{\partial x_i} \frac{\partial u_k}{\partial x_i}} - \nu \frac{\partial \epsilon}{\partial x_k} \right) \\ &\quad \{1\} \qquad \qquad \{1\} \qquad \qquad \{Re_l^{-1}\} \\ &- 2 \nu \frac{\partial U_i}{\partial x_k} \left( \overline{\frac{\partial u_i}{\partial x_1} \frac{\partial u_k}{\partial x_1}} + \overline{\frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_k}} \right) - 2 \nu \overline{u_k \frac{\partial u_i}{\partial x_1} \frac{\partial^2 U_i}{\partial x_k \partial x_1}} \\ &\quad \{Re_l^{-1/2}\} \qquad \qquad \{Re_l^{-1/2}\} \\ &- 2 \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_1} \frac{\partial u_k}{\partial x_1}} - 2 \overline{\left( \nu \frac{\partial^2 u_i}{\partial x_k \partial x_1} \right)^2} \\ &\quad \{Re_l^{+1/2}\} \qquad \qquad \{Re_l^{+1/2}\} \end{aligned}$$

ENERGY DISSIPATION RATE AT HIGH REYNOLDS NUMBERS

$$\frac{\partial \epsilon}{\partial t} + U_k \frac{\partial \epsilon}{\partial x_k} = \underbrace{- \frac{\partial}{\partial x_k} \left( \overline{\nu u_k \frac{\partial u_i}{\partial x_1} \frac{\partial u_i}{\partial x_1}} + 2 \overline{\frac{\nu}{\rho} \frac{\partial p}{\partial x_i} \frac{\partial u_k}{\partial x_i}} \right)}_{(A)} - \underbrace{2 \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_1} \frac{\partial u_k}{\partial x_1}}}_{(B)} - \underbrace{2 \overline{\left( \nu \frac{\partial^2 u_i}{\partial x_k \partial x_1} \right)^2}}_{(C)}$$

Terms (B) and (C) individually vary as  $Re_1^{1/2}$ , i.e. become infinite, but when taken together are of order unity. When these terms are modelled they must be modelled together! However, the  $\epsilon$  equation is in general modelled using a standard type of convection/diffusion equation assuming isotropy:

$$\begin{aligned} \frac{\partial \epsilon}{\partial t} + U_k \frac{\partial \epsilon}{\partial x_k} &= \frac{\partial}{\partial x_k} \left( (\nu + \nu_t / \sigma_\epsilon) \frac{\partial \epsilon}{\partial x_k} \right) + c_{\epsilon 1} \frac{\epsilon}{k} P_k - c_{\epsilon 2} \frac{\epsilon^2}{k} \\ c_\epsilon &= 1.3 \qquad c_{\epsilon 1} = 1.45 \qquad c_{\epsilon 2} = 1.92 \qquad \sigma_\epsilon = 1.3 \end{aligned}$$

HIGH REYNOLDS NO. k-ε MODEL (incompressible flow)

$$\overline{u_i u_j} = -\nu_t \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \frac{2}{3} \delta_{ij} k$$

$$\nu_t = c_\mu k^2 / \varepsilon$$

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k} \left( (\nu + \nu_t / \sigma_k) \frac{\partial k}{\partial x_k} \right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k} \left( (\nu + \nu_t / \sigma_\varepsilon) \frac{\partial \varepsilon}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

$$P_k = - \overline{u_i u_k} \frac{\partial U_i}{\partial x_k}$$

$$c_\mu = 0.09 \quad c_{\varepsilon 1} = 1.44 \quad c_{\varepsilon 2} = 1.92 \quad \sigma_k = 1.0 \quad \sigma_\varepsilon = 1.3$$

LOW REYNOLDS NO. k-ε MODEL (incompressible flow)

$$\frac{\partial \tilde{\varepsilon}}{\partial t} + U_k \frac{\partial \tilde{\varepsilon}}{\partial x_k} = \frac{\partial}{\partial x_k} \left( (\nu + \nu_t / \sigma_\varepsilon) \frac{\partial \tilde{\varepsilon}}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k} + 2\nu\nu_t \left( \frac{\partial^2 U_i}{\partial x_k \partial x_1} \right)^2$$

$$\varepsilon = \tilde{\varepsilon} + 2 \nu \left( \frac{\partial \sqrt{k}}{\partial x_j} \right)^2$$

$$\nu_t = c_\mu k^2 / \tilde{\varepsilon}$$

$$c_\mu = 0.09 \exp \left[ -3.4 / (1 + R_t / 50)^2 \right]$$

$$c_{\varepsilon 2} = 1.92 (1 - 0.3 \exp(-R_t^2))$$

$$R_t = k^2 / (\nu \tilde{\varepsilon})$$

The other equations are the same as in the high Reynolds no. k-ε model.

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = P_{ij} + D_{ij} + \Psi_{ij} - \epsilon_{ij}$$

$P_{ij}$  Production by mean strain rate, requires no modelling:

$$P_{ij} = - \left( \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} + \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} \right)$$

$D_{ij}$  Diffusion, can be modelled using a gradient diffusion approximation:

$$D_{ij} = \frac{\partial}{\partial x_k} \left( c_s \frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial}{\partial x_l}(\overline{u_i u_j}) \right)$$

A model constant  $c_s = 0.22$  has been introduced. The following simpler model is also used:

$$D_{ij} = \frac{\partial}{\partial x_k} \left( \nu_t \frac{\partial}{\partial x_k}(\overline{u_i u_j}) \right)$$

Instead of  $\nu_t$  only,  $(\nu + \nu_t)$  can also be used.

$\Psi_{ij}$  The pressure-strain relation can be modelled by:

$$\Psi_{ij} = - c_1 \frac{\epsilon}{k} \left( \overline{u_i u_j} - \frac{2}{3} k \delta_{ij} \right) - c_2 \left( P_{ij} - \frac{2}{3} P_k \delta_{ij} \right)$$

$$P_k = \frac{1}{2} P_{ii}$$

$\epsilon_{ij}$  Viscous destruction, can be modelled by (assuming isotropy):

$$\epsilon_{ij} = \frac{2}{3} \delta_{ij} \epsilon$$

The turbulent kinetic energy is calculated using the standard model equation:

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k} \left( (\nu + \nu_t / \sigma_k) \frac{\partial k}{\partial x_k} \right) + P_k - \varepsilon$$

The energy dissipation rate can be calculated using the model equation from the k-ε model:

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k} \left( (\nu + \nu_t / \sigma_\varepsilon) \frac{\partial \varepsilon}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

or, using the information about the anisotropy in the flow, the following equation can be used:

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k} \left( c_\varepsilon \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

The model constants are:

$$\begin{aligned} c_s &= 0.22 \\ c_1 &= 1.80 \\ c_2 &= 0.60 \\ c_\varepsilon &= 0.18 \\ c_{\varepsilon 1} &= 1.44 \\ c_{\varepsilon 2} &= 1.92 \end{aligned}$$

MODEL EQUATION EQUATIONS ASM-MODEL (incompressible flow)  
 (Algebraic stress turbulence model)

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = P_{ij} + D_{ij} + \Psi_{ij} - \epsilon_{ij}$$

This equation can be simplified by combining the convection and diffusion terms:

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + \left[ U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) + \frac{\partial}{\partial x_k} \left( \nu_t \frac{\partial}{\partial x_k}(\overline{u_i u_j}) \right) \right] = P_{ij} + \Psi_{ij} - \epsilon_{ij}$$

which leads to:

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + T_{ij} = P_{ij} + \Psi_{ij} - \epsilon_{ij}$$

This equation can be simplified using Rodi's approximation. It is assumed that the transport of  $\overline{u_i u_j}$  is proportional to the transport of  $k$ , with the ratio  $\overline{u_i u_j}/k$  as proportionality factor:

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + T_{ij} = \frac{\overline{u_i u_j}}{k} \left( \frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} - \text{Diff}(k) \right) = \frac{\overline{u_i u_j}}{k} (P_k - \epsilon)$$

This results in the following algebraic equation for the Reynolds stresses:

$$\overline{u_i u_j} = \frac{2}{3} k \delta_{ij} \left[ 1 - \frac{(1-c_2) \frac{P_k}{\epsilon}}{\frac{P_k}{k} - (1-c_1)} \right] + \frac{1-c_2}{\frac{P_k}{k} - (1-c_1)} \frac{k}{\epsilon} \left( P_{ij} - \frac{2}{3} P_k \delta_{ij} + \frac{c_3}{1-c_2} A_{ij} \right)$$

The term  $A_{ij}$  is the so-called added convection term, which is zero when the equations are written out in cartesian coordinates but plays an important role when the equations are written in cylindrical-polar coordinates.

The equations used for calculating  $k$  and  $\epsilon$  are the same as in the Reynolds stress model.

The model constants are:

$$c_1 = 2.50$$

$$c_2 = 0.55$$

VECTOR NOTATION

$$\vec{X} \cdot \vec{Y} = x_i y_i$$

(Scalar/dot product)

$$\boxed{\phantom{0}} \cdot \boxed{\phantom{0}} = \boxed{\phantom{0}}$$

$$\vec{X}\vec{Y} = \vec{Z}$$

(Dyadic product)  $z_{ij} = x_i y_j$

$$\boxed{\phantom{0}} \boxed{\phantom{0}} = \boxed{\phantom{00}}$$

$$\vec{A} \cdot \vec{B} = \vec{C}$$

$c_{ij} = a_{ik} b_{kj}$

$$\boxed{\phantom{00}} \cdot \boxed{\phantom{00}} = \boxed{\phantom{00}}$$

$$\vec{A} : \vec{B} = a_{ij} b_{ji}$$

(Double dotted product)

$$\boxed{\phantom{00}} : \boxed{\phantom{00}} = \boxed{\phantom{0}}$$