OVERVIEW BALANCE EQUATIONS ISOTHERMAL SINGLE-PHASE FLOW AND TURBULENCE MODELS

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CONTENTS

2 OVERVIEW EQUATIONS FOR TIME AVERAGED INCOMPRESSIBLE STEADY STATE FLOW

3 CONTINUITY EQUATION

MOMENTUM BALANCE

MEAN MOTION - CONTINUITY

MEAN MOTION - MOMENTUM

4 DEFINITIONS REYNOLDS AVERAGING

5 REYNOLDS STRESSES - ANALYTICAL EQUATION (incompressible flow)

6 TURBULENT KINETIC ENERGY - ANALYTICAL EQUATION (Incompressible flow)

TURBULENT KINETIC ENERGY - MODEL EQUATION (Incompressible flow)

7 ENERGY DISSIPATION RATE - ANALYTICAL EQUATION

ENERGY DISSIPATION RATE AT HIGH REYNOLDS NUMBERS

8 HIGH REYNOLDS NO. k-ε MODEL (incompressible flow)

LOW REYNOLDS NO. k-ε MODEL (incompressible flow)

9 MODEL EQUATIONS FULL RSM-MODEL (incompressible flow) - PAGE A

10 MODEL EQUATIONS FULL RSM-MODEL (incompressible flow) - PAGE B

11 MODEL EQUATIONS ASM-MODEL (incompressible flow)

12 VECTOR NOTATION

OVERVIEW EQUATIONS FOR TIME AVERAGED INCOMPRESSIBLE STEADY STATE FLOW

The continuity equation:

$$\vec{\nabla}.\vec{U} = 0$$

The momentum balance:

$$\vec{\nabla}.(\vec{U}\vec{U}) = \vec{\nabla}.(\vec{\vec{\pi}} + \vec{\vec{\pi}}_T)$$

The molecular stress tensor:

$$\vec{\vec{\pi}} = -\frac{P}{\rho}\vec{\vec{I}} + \nu(\vec{\nabla}\vec{U} + (\vec{\nabla}\vec{U})^T)$$

The Reynolds stress tensor:

$$\vec{\vec{\pi}}_T = -\overline{\vec{u}\vec{u}}$$

k-ε model:

$$\vec{\vec{\pi}}_T = -\frac{2}{3}k\vec{\vec{I}} + \nu_t(\vec{\nabla}\vec{U} + (\vec{\nabla}\vec{U})^T)$$

Turbulent viscosity:

$$\nu_t = c_\mu k^2/\varepsilon$$

Model equation for k:

$$\vec{U}.\vec{\nabla}k = \vec{\nabla}.((\nu + \nu_t/\sigma_k)\vec{\nabla}k) + P_k - \varepsilon$$

$$P_k = -\overline{\vec{u}\vec{u}}:(\vec{\nabla}\vec{U})$$

Model equation for ε:

$$\vec{U}.\vec{\nabla}\varepsilon = \vec{\nabla}.((\nu + \nu_t/\sigma_\varepsilon)\vec{\nabla}\varepsilon) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

Algebraic stress model:

$$\vec{\vec{\pi}}_T = \frac{2}{3}k\vec{\vec{I}}\left[1 - \frac{(1-c_2)\frac{P_k}{\varepsilon}}{\frac{P_k}{\varepsilon}-(1-c_1)}\right] + \frac{1-c_2}{\frac{P_k}{\varepsilon}-(1-c_1)}\frac{k}{\varepsilon}\left(\vec{\vec{P}} - \frac{2}{3}P_k\vec{\vec{I}}\right)$$

$$\vec{\vec{P}} = -\left(\overline{\vec{u}\vec{u}}.(\vec{\nabla}\vec{U}) + (\overline{\vec{u}\vec{u}}.(\vec{\nabla}\vec{U}))^T\right)$$

CONTINUITY EQUATION

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\hat{\rho}\ \hat{U}_i) = 0$$

MOMENTUM BALANCE

$$\frac{\partial}{\partial t}(\hat{\rho}\hat{U}_i) + \frac{\partial}{\partial x_k}(\hat{\rho}\hat{U}_i\hat{U}_k) = -\frac{\partial \hat{P}}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\mu(\frac{\partial \hat{U}_i}{\partial x_k} + \frac{\partial \hat{U}_k}{\partial x_i} - \frac{2}{3}\frac{\partial \hat{U}_m}{\partial x_m}\delta_{ik})\right) + \hat{\rho}g_i$$

$$\frac{2}{3}\frac{\partial \hat{U}_m}{\partial x_m}\delta_{ik} = 0 \qquad \text{for incompressible flows}$$

MEAN MOTION - CONTINUITY

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho\ U_i) = 0$$

For incompressible flows:

$$\frac{\partial U_i}{\partial x_i} = 0$$

MEAN MOTION - MOMENTUM

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho U_i U_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\mu\frac{\partial U_k}{\partial x_k} - \mu\ \frac{2}{3}\frac{\partial U_m}{\partial x_m}\delta_{ik} - \rho\overline{u_i u_k} \right) + \rho g_i$$

For incompressible flows:

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\nu\frac{\partial U_k}{\partial x_k} - \overline{u_i u_k} \right) + g_i$$

DEFINITIONS REYNOLDS AVERAGING

$$\hat{U}_i = U_i + u_i$$

$$\overline{\hat{U}_i \hat{U}_j} = U_i U_j + \overline{u_i u_j}$$

$$\overline{\hat{U}_i} = U_i$$

$$\overline{u_i} = 0$$

$$\hat{\Phi} = \Phi + \varphi$$

$$\overline{\hat{\Phi}} = \Phi$$

$$\overline{\varphi} = 0$$

$$\overline{\hat{\Phi}\, \hat{U}_i} = \Phi\, U_i + \overline{\varphi\, u_j}$$

$$\overline{\hat{\rho}} = \rho$$

$$\hat{P} = P + p$$

REYNOLDS STRESSES - ANALYTICAL EQUATION (incompressible flow)

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = - \underbrace{\left(\overline{u_j u_k} \frac{\partial U_i}{\partial x_k} + \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} \right)}_{(A)}$$

$$- \underbrace{\frac{\partial}{\partial x_k} \left(\overline{u_i u_j u_k} - \nu \frac{\partial}{\partial x_k}(\overline{u_i u_j}) + \overline{\frac{p}{\rho} (u_i \delta_{jk} + u_j \delta_{ik})} \right)}_{(B)}$$

$$+ \underbrace{\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}}_{(C)} - \underbrace{2 \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}}_{(D)}$$

(A) Rate of generation of $\overline{u_i u_j}$ by the effects of the mean strain. Transfer of energy from the mean flow to turbulence. Symbol P_{ij}.

(B) "Diffusion" D_{ij} of $\overline{u_i u_j}$ in three components:

(1) "turbulent" diffusion involving triple products.

(2) "viscous" diffusion written in this way for convenience.

(3) "pressure" diffusion.

(C) The pressure-strain correlation. Symbol Ψ_{ij}. Appears by rearrangement of the pressure terms:

$$\overline{u_j \frac{\partial}{\partial x_i}(P+p)} = \overline{u_j \frac{\partial p}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{(pu_j)} - \overline{p \frac{\partial u_j}{\partial x_i}} = \frac{\partial}{\partial x_k}\overline{(pu_j)} - \overline{p \frac{\partial u_j}{\partial x_i}}$$

This decomposition into pressure diffusion plus pressure strain is convenient because the second component has zero trace on contraction. It is not unique.

(D) Viscous destruction of $\overline{u_i u_j}$. Symbol ε_{ij}. Appears by rearrangement of the viscous terms.

$$\overline{\nu u_j \frac{\partial^2}{\partial x_k^2}(U_i + u_i)} + \overline{\nu u_i \frac{\partial^2}{\partial x_k^2}(U_j + u_j)} = \nu \frac{\partial^2}{\partial x_k^2}\overline{(u_i u_j)} - 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}$$

Convenient because the first component has the form of viscous gradient diffusion of $\overline{u_i u_j}$. The second component has useful properties.

TURBULENT KINETIC ENERGY - ANALYTICAL EQUATION (Incompressible flow)

$$k = \frac{1}{2} \overline{u_i u_i}$$

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \underbrace{- \overline{u_i u_k} \frac{\partial U_i}{\partial x_k}}_{(A)} \underbrace{- \frac{\partial}{\partial x_k}\left(\frac{1}{2} \overline{u_k u_i^2} - \nu \frac{\partial k}{\partial x_k} + \overline{\frac{p}{\rho} u_k} \right)}_{(B)} \underbrace{- \nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k}}}_{(C)}$$

(A) Production rate of k by interaction with the mean strain rate. Symbol P_k.

(B) "Diffusion" of k: (1) "Turbulent", (2) "Viscous", (3) "Pressure"

(C) Viscous destruction of k, symbol ε.

TURBULENT KINETIC ENERGY - MODEL EQUATION (Incompressible flow)

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k}\left((\nu + \nu_t/\sigma_k) \frac{\partial k}{\partial x_k} \right) + P_k - \varepsilon$$

$$P_k = - \overline{u_i u_k} \frac{\partial U_i}{\partial x_k}$$

$$\nu_t = c_\mu k^2/\varepsilon$$

$$c_\mu = 0.09$$

$$\sigma_k = 1.0$$

This model equation is used in both the low Reynolds no. and the high Reynolds no. k-ε eddy viscosity model, as well as in higher order closure models like the ASM model and full Reynolds stress models.

ENERGY DISSIPATION RATE - ANALYTICAL EQUATION

$$\varepsilon = \nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}}$$

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = -\frac{\partial}{\partial x_k}\left(\nu \overline{u_k \frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}} + 2\frac{\nu}{\rho}\overline{\frac{\partial p}{\partial x_i}\frac{\partial u_k}{\partial x_i}} - \nu \frac{\partial \varepsilon}{\partial x_k}\right)$$

$$\{1\} \qquad \{1\} \qquad \{Re_l^{-1}\}$$

$$- 2\nu \frac{\partial U_i}{\partial x_k}\left(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}}\right) - 2\nu \overline{u_k \frac{\partial u_i}{\partial x_l}}\frac{\partial^2 U_i}{\partial x_k \partial x_l}$$

$$\{Re_l^{-1/2}\} \qquad \{Re_l^{-1/2}\}$$

$$- 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} - 2\overline{\left(\nu \frac{\partial^2 u_i}{\partial x_k \partial x_l}\right)^2}$$

$$\{Re_l^{+1/2}\} \qquad \{Re_l^{+1/2}\}$$

ENERGY DISSIPATION RATE AT HIGH REYNOLDS NUMBERS

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \underbrace{-\frac{\partial}{\partial x_k}\left(\nu \overline{u_k \frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}} + 2\frac{\nu}{\rho}\overline{\frac{\partial p}{\partial x_i}\frac{\partial u_k}{\partial x_i}}\right)}_{(A)} \underbrace{- 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}}}_{(B)} \underbrace{- 2\overline{\left(\nu \frac{\partial^2 u_i}{\partial x_k \partial x_l}\right)^2}}_{(C)}$$

Terms (B) and (C) individually vary as $Re_l^{1/2}$, i.e. become infinite, but when taken together are of order unity. When these terms are modelled they must be modelled together! However, the ε equation is in general modelled using a standard type of convection/diffusion equation assuming isotropy:

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k}\left((\nu + \nu_t/\sigma_\varepsilon) \frac{\partial \varepsilon}{\partial x_k}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k} P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$c_\varepsilon = 1.3 \qquad c_{\varepsilon 1} = 1.45 \qquad c_{\varepsilon 2} = 1.92 \qquad \sigma_\varepsilon = 1.3$

HIGH REYNOLDS NO. k-ε MODEL (incompressible flow)

$$\overline{u_i u_j} = -\nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \frac{2}{3} \delta_{ij} k$$

$$\nu_t = c_\mu k^2/\varepsilon$$

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k}\left((\nu + \nu_t/\sigma_k) \frac{\partial k}{\partial x_k} \right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k}\left((\nu + \nu_t/\sigma_\varepsilon) \frac{\partial \varepsilon}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

$$P_k = - \overline{u_i u_k} \frac{\partial U_i}{\partial x_k}$$

$c_\mu = 0.09$ $\quad c_{\varepsilon 1} = 1.44$ $\quad c_{\varepsilon 2} = 1.92$ $\quad \sigma_k = 1.0$ $\quad \sigma_\varepsilon = 1.3$

LOW REYNOLDS NO. k-ε MODEL (incompressible flow)

$$\frac{\partial \tilde{\varepsilon}}{\partial t} + U_k \frac{\partial \tilde{\varepsilon}}{\partial x_k} = \frac{\partial}{\partial x_k}\left((\nu + \nu_t/\sigma_\varepsilon) \frac{\partial \tilde{\varepsilon}}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k} + 2\nu\nu_t \left(\frac{\partial^2 U_i}{\partial x_k \partial x_l} \right)^2$$

$$\varepsilon = \tilde{\varepsilon} + 2 \nu \left(\frac{\partial \sqrt{k}}{\partial x_j} \right)^2$$

$$\nu_t = c_\mu k^2/\tilde{\varepsilon}$$

$$c_\mu = 0.09 \exp\left[-3.4/(1 + R_t/50)^2 \right]$$

$$c_{\varepsilon 2} = 1.92 (1 - 0.3 \exp(-R_t^2))$$

$$R_t = k^2/(\nu \tilde{\varepsilon})$$

The other equations are the same as in the high Reynolds no. k-ε model.

MODEL EQUATION EQUATIONS FULL RSM-MODEL (incompressible flow) - PAGE A
(Reynolds stress model)

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = P_{ij} + D_{ij} + \Psi_{ij} - \varepsilon_{ij}$$

P_{ij} Production by mean strain rate, requires no modelling:

$$P_{ij} = - \left(\overline{u_j u_k} \frac{\partial U_i}{\partial x_k} + \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} \right)$$

D_{ij} Diffusion, can be modelled using a gradient diffusion approximation:

$$D_{ij} = \frac{\partial}{\partial x_k} \left(c_s \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial}{\partial x_l}(\overline{u_i u_j}) \right)$$

A model constant $c_s = 0.22$ has been introduced. The following simpler model is also used:

$$D_{ij} = \frac{\partial}{\partial x_k} \left(\nu_t \frac{\partial}{\partial x_k}(\overline{u_i u_j}) \right)$$

Instead of ν_t only, $(\nu + \nu_t)$ can also be used.

Ψ_{ij} The pressure-strain relation can be modelled by:

$$\Psi_{ij} = - c_1 \frac{\varepsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} k \, \delta_{ij} \right) - c_2 \left(P_{ij} - \frac{2}{3} P_k \, \delta_{ij} \right)$$

$$P_k = \frac{1}{2} P_{ii}$$

ε_{ij} Viscous destruction, can be modelled by (assuming isotropy):

$$\varepsilon_{ij} = \frac{2}{3} \delta_{ij} \, \varepsilon$$

MODEL EQUATION EQUATIONS FULL RSM-MODEL (incompressible flow) - PAGE B

The turbulent kinetic energy is calculated using the standard model equation:

$$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k}\left((\nu + \nu_t/\sigma_k) \frac{\partial k}{\partial x_k} \right) + P_k - \varepsilon$$

The energy dissipation rate can be calculated using the model equation from the k-ε model:

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k}\left((\nu + \nu_t/\sigma_\varepsilon) \frac{\partial \varepsilon}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

or, using the information about the anisotropy in the flow, the following equation can be used:

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k}\left(c_\varepsilon \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_k} \right) + c_{\varepsilon 1} \frac{\varepsilon}{k} P_k - c_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

The model constants are:

$c_s = 0.22$
$c_1 = 1.80$
$c_2 = 0.60$
$c_\varepsilon = 0.18$
$c_{\varepsilon 1} = 1.44$
$c_{\varepsilon 2} = 1.92$

MODEL EQUATION EQUATIONS ASM-MODEL (incompressible flow)
(Algebraic stress turbulence model)

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) = P_{ij} + D_{ij} + \Psi_{ij} - \varepsilon_{ij}$$

This equation can be simplified by combining the convection and diffusion terms:

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + \left[U_k \frac{\partial}{\partial x_k}(\overline{u_i u_j}) + \frac{\partial}{\partial x_k}\left(\nu_t \frac{\partial}{\partial x_k}(\overline{u_i u_j})\right) \right] = P_{ij} + \Psi_{ij} - \varepsilon_{ij}$$

which leads to:

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + T_{ij} = P_{ij} + \Psi_{ij} - \varepsilon_{ij}$$

This equation can be simplified using Rodi's approximation. It is assumed that the transport of $\overline{u_i u_j}$ is proportional to the transport of k, with the ratio $\overline{u_i u_j}/k$ as proportionality factor:

$$\frac{\partial}{\partial t}(\overline{u_i u_j}) + T_{ij} = \frac{\overline{u_i u_j}}{k}\left(\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} - \mathrm{Diff}(k)\right) = \frac{\overline{u_i u_j}}{k}(P_k - \varepsilon)$$

This results in the following algebraic equation for the Reynolds stresses:

$$\overline{u_i u_j} = \frac{2}{3} k \,\delta_{ij} \left[1 - \frac{(1-c_2)\frac{P_k}{\varepsilon}}{\frac{P_k}{\varepsilon}-(1-c_1)} \right] + \frac{1-c_2}{\frac{P_k}{\varepsilon}-(1-c_1)} \frac{k}{\varepsilon}\left(P_{ij} - \frac{2}{3} P_k \,\delta_{ij} + \frac{c_3}{1-c_2} A_{ij}\right)$$

The term A_{ij} is the so-called added convection term, which is zero when the equations are written out in cartesian coordinates but plays an important role when the equations are written in cylindrical-polar coordinates.

The equations used for calculating k and ε are the same as in the Reynolds stress model.

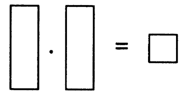
The model constants are:

$c_1 = 2.50$
$c_2 = 0.55$

VECTOR NOTATION

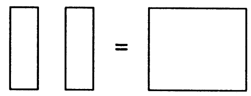
$\vec{X}.\vec{Y} = x_i y_i$

(Scalar/dot product)

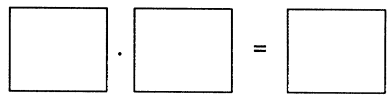


$\vec{X}\vec{Y} = \vec{\vec{Z}}$ $\quad z_{ij} = x_i y_j$

(Dyadic product)



$\vec{\vec{A}}.\vec{\vec{B}} = \vec{\vec{C}}$ $\quad c_{ij} = a_{ik} b_{kj}$



$\vec{\vec{A}}:\vec{\vec{B}} = a_{ij} b_{ji}$

(Double dotted product)